

ARNOLD DIFFUSION IN NEARLY INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT. In this paper, Arnold diffusion is proved to be generic phenomenon in nearly integrable convex Hamiltonian systems with three degrees of freedom:

$$H(x, y) = h(y) + \epsilon P(x, y), \quad x \in \mathbb{T}^3, \ y \in \mathbb{R}^3.$$

Under typical perturbation ϵP , the system admits “connecting” orbit that passes through any two prescribed small balls in the same energy level $H^{-1}(E)$ provided E is bigger than the minimum of the average action, namely, $E > \min \alpha$.

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1. INTRODUCTION

For nearly integrable Hamiltonian systems, the set of KAM tori has a relatively large Lebesgue measure in phase space. For systems with two degrees of freedom, it implies the dynamical stability: all orbits are stable, the variation of actions stays small for all the time as each 2-dimensional KAM torus separates the 3-dimensional energy level. However, this is a special property of lower-dimensional space, KAM

torus of n -dimension does not separate $(2n - 1)$ -dimensional energy level if $n > 2$. It is conceivable that the complement of all n -dimensional invariant tori forms dense and connected set in phase space. This would mean that by arbitrary small changes of the initial states one would find orbits along which the action variables ultimately escape. The underlying phenomenon is now called “Arnold diffusion”.

Conjecture ([Ar2, AKN]): *The typical case in a higher-dimensional problems is topological instability: through an arbitrarily small neighborhood of any point there passes a phase trajectories along which the slow variables drift away from the initial value by a quantity of order 1.*

Since the celebrated example of Arnold [Ar1] was published half a century ago, there are many works for the study of this problem. In recent years, it has become clear that diffusion is a typical phenomenon in so called *a priori* unstable systems, refer to [Be3, CY1, CY2, DLS, LC, Tr]. The *a priori* unstable condition guarantees the existence of normally hyperbolic cylinder, from which one derives certain regularity of the barrier functions. The genericity of the diffusion is obtained by using the regularity [CY1, CY2]. There are also many works for the study of the problem, for instance, see [Bs, BCV, FM, DH1, DH2, GL, GR1, GR2, KL1, KL2, X, Zha].

General perturbation of integrable Hamiltonian is usually called *a priori* stable system. A bit away from strong complete resonance in such systems, some pieces of normally hyperbolic cylinder still exist and the method for *a priori* unstable system can also be applied [BKZ, Be4]. For *a priori* stable systems with three degrees of freedom, a notable difficulty occurs at the point of double resonance, around which the cylinder for prescribed single resonance may disappear. The averaged system has two homoclinic orbits associated with different classes in $H_1(\mathbb{T}^2, \mathbb{Z})$. As the energy decreases, the periodic orbit on the cylinder simultaneously approach to these two homoclinic orbits. Correspondingly, the transition chain in $H^1(\mathbb{T}^2, \mathbb{R})$ for the single resonance may break. To solve this difficulty, Mather suggested a path in $H^1(\mathbb{T}^2, \mathbb{R})$ to cross double resonance, along which one moves the cohomology class in the channel determined by the prescribed homology class and switches it to the channel determined by one of these two classes when it is getting close to the double resonance [Ma4, Ma6, Ma7].

In this paper, the path we choose to construct transition chain is different from that suggested by Mather. We find an annulus surrounding the flat of double resonance in $H^1(\mathbb{T}^2, \mathbb{R})$, which has a foliation of circles. Each of these circles is actually a transition chains of incomplete intersection. Although the annulus is not so thick, each single resonance path extends into. It allows us to use one of these circles connecting one single resonance path to another. In this way, we find a path of transition chain along which the diffusion orbits are constructed by variational method.

1.1. Statement of the main result. We consider nearly integrable Hamiltonian systems with 3 degrees of freedom:

$$(1.1) \quad H(x, y) = h(y) + \epsilon P(x, y), \quad (x, y) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

where h is assumed to strictly convex, namely, the Hessian matrix $\partial^2 h / \partial y^2$ is positive definite. It is also assumed that $\min h = 0$, both h and P are C^r -function with $r \geq 5$.

For $E > 0$, let $H^{-1}(E) = \{(x, y) : H(x, y) = E\}$ denote the energy level set, $B \subset \mathbb{R}^3$ denote a ball in \mathbb{R}^3 such that $\bigcup_{E' \leq E+1} h^{-1}(E') \subset B$. Let $\mathfrak{S}_a, \mathfrak{B}_a \subset C^r(\mathbb{T}^3 \times B)$ denote a sphere and a ball with radius $a > 0$ respectively: $F \in \mathfrak{S}_a$ if and only $\|F\|_{C^r} = a$ and $F \in \mathfrak{B}_a$ if and only $\|F\|_{C^r} \leq a$. They inherit the topology from $C^r(\mathbb{T}^3 \times B)$.

Let \mathfrak{R}_a be a set residual in \mathfrak{S}_a , each $P \in \mathfrak{R}_a$ is associated with a set R_P residual in the interval $[0, a_P]$ with $a_P \leq a$. A set \mathfrak{C}_a is said cusp-residual in \mathfrak{B}_a if

$$\mathfrak{C}_a = \{\lambda P : P \in \mathfrak{R}_a, \lambda \in R_P\}.$$

Let Φ_H^t denote the Hamiltonian flow determined by H . Given an initial value (x, y) , $\Phi_H^t(x, y)$ generates an orbit of the Hamiltonian flow $(x(t), y(t))$. An orbit $(x(t), y(t))$ is said to visit $B_\delta(y_0) \subset \mathbb{R}^3$ if there exists $t \in \mathbb{R}$ such that $y(t) \in B_\delta(y_0)$ a ball centered at y_0 with radius δ .

The main result of this paper is the following:

Theorem 1.1. *Given any two balls $B_\delta(x_0, y_0), B_\delta(x_k, y_k) \subset \mathbb{T}^3 \times \mathbb{R}^3$ and finitely many small balls $B_\delta(y_i) \subset \mathbb{R}^3$ ($i = 0, 1, \dots, k$), where $y_i \in h^{-1}(E)$ with $E > 0$ and $\delta > 0$ is small, there exists a cusp-residual set $\mathfrak{P}_{\epsilon_0}$ such that for each $\epsilon P \in \mathfrak{P}_{\epsilon_0}$, the Hamiltonian flow Φ_H^t admits orbits that, on the way between passing through $B_\delta(x_0, y_0)$ and $B_\delta(x_k, y_k)$, visit the balls $B_\delta(y_i)$ in turn.*

Remark. The result proved here is stronger than what was formulated in [Ar2]. By dropping the requirement that orbit passes two prescribed balls in the phase space and using the same construction, one can get an orbit that visits these balls $B_\delta(y_i) \subset \mathbb{R}^3$ ($i = 0, 1, \dots, k$) infinitely many times with any prescribed order, as it was announced in [Ma4]. Indeed, as finitely many balls are given, there exists a path with finite length passing through finitely many resonance layers and connecting any two of these balls directly. It does not damage the cusp-residual property.

Using the same method, one can prove the same result for time-periodic systems with two degrees of freedom. The statement of the result will be: for typical time-periodic perturbations of integrable Hamiltonian with 2-degrees of freedom, the Hamiltonian flow admits orbits passing through any prescribed two balls $B_\delta(x_0, y_0)$ and $B_\delta(x_k, y_k)$ in the phase space and finitely many small balls $B_\delta(y_i) \subset \mathbb{R}^2$ ($i = 0, 1, \dots, k$) in the action variable space. The proof is a easier from technical point of view, one can see it in the following proof.

1.2. Brief introduction of Mather theory. We use variational method to prove the result, which is based on Mather theory. This theory is established for Tonelli Lagrangian.

Definition 1.1. *Let M be a closed manifold. A C^2 -function $L: TM \times \mathbb{T} \rightarrow \mathbb{R}$ is called Tonelli Lagrangian if it satisfies the following conditions:*

POSITIVE DEFINITENESS. *For each $(x, t) \in M \times \mathbb{T}$, the Lagrangian function is strictly convex in velocity: the Hessian $\partial_{\dot{x}\dot{x}}L$ is positive definite.*

SUPER-LINEAR GROWTH. *We assume that L has fiber-wise superlinear growth: for each $(x, t) \in M \times \mathbb{T}$, we have $L/\|\dot{x}\| \rightarrow \infty$ as $\|\dot{x}\| \rightarrow \infty$.*

COMPLETENESS. *All solutions of the Lagrangian equations are well defined for all $t \in \mathbb{R}$.*

For autonomous case, the completeness is automatically satisfied, since each orbit entirely stays in certain compact energy surface.

Let $\eta_c(x)$ denote a closed 1-form $\langle \eta_c(x), dx \rangle$ evaluated at x , with its first co-homology class $[\langle \eta_c(x), dx \rangle] = c \in H^1(M, \mathbb{R})$. We introduce a Lagrange multiplier $\eta_c = \langle \eta_c(x), \dot{x} \rangle$. Without danger of confusion, we call it closed 1-form also.

For each C^1 curve $\gamma: \mathbb{R} \rightarrow M$ with period k , there is unique probability measure μ_γ on $TM \times \mathbb{T}$ so that the following holds

$$\int_{TM \times \mathbb{T}} f d\mu_\gamma = \frac{1}{k} \int_0^k f(d\gamma(s), s) ds$$

for each $f \in C^0(TM \times \mathbb{T}, \mathbb{R})$, where we use the notation $d\gamma = (\gamma, \dot{\gamma})$. Let

$$\Gamma = \{\mu_\gamma \mid \gamma \in C^1(\mathbb{R}, M) \text{ is periodic of } k\}.$$

The set \mathfrak{H} of holonomic probability measures is the closure of Γ in the vector space of continuous linear functionals. One can see that \mathfrak{H} is convex.

For each $\nu \in \mathfrak{H}$ the action $A_c(\nu)$ is defined as follows

$$A_c(\nu) = \int (L - \eta_c) d\nu.$$

It is proved in [Ma1, Me] that for each co-homology class c there exists at least one invariant probability measure μ_c minimizing the action over \mathfrak{H}

$$A_c(\mu_c) = \inf_{\nu \in \mathfrak{H}} \int (L - \eta_c) d\nu,$$

called c -minimal measure. Let $\mathfrak{H}_c \subset \mathfrak{H}$ be the set of c -minimal measures, the Mather set $\tilde{\mathcal{M}}(c)$ is defined as

$$\tilde{\mathcal{M}}(c) = \bigcup_{\mu_c \in \mathfrak{H}_c} \text{supp} \mu_c.$$

The α -function is defined as $\alpha(c) = -A_c(\mu_c) : H^1(M, \mathbb{R}) \rightarrow \mathbb{R}$, it is convex, finite everywhere with super-linear growth. Its Legendre transformation $\beta : H_1(M, \mathbb{R}) \rightarrow \mathbb{R}$ is called β -function

$$\beta(\omega) = \max_c (\langle \omega, c \rangle - \alpha(c)).$$

It is also convex, finite everywhere with super-linear growth (see [Ma1]).

Note that $\int \lambda d\mu_\gamma = 0$ for each exact 1-form λ and each $\mu_\gamma \in \Gamma$. Thus, for each measure $\mu \in \mathfrak{H}$ one can define its rotation vector $\omega(\mu) \in H_1(M, \mathbb{R})$ such that

$$\langle [\lambda], \omega(\mu) \rangle = \int \lambda d\mu,$$

for every closed 1-form λ on M . For a closed curve $\gamma: [0, k] \rightarrow M$ its rotation vector is defined as

$$[\gamma] = \frac{\bar{\gamma}(k) - \bar{\gamma}(0)}{k},$$

where $\bar{\gamma}$ stands for the lift of γ to the universal covering \mathbb{R}^n . Let $\gamma_k: [0, k] \rightarrow M$ be a closed curve such that $[\gamma_k] = \omega_k$ and

$$\frac{1}{k}A(\gamma_k) = \inf_{[\gamma] = \omega_k} \frac{1}{k} \int_0^k L(d\gamma(t), t) dt.$$

Obviously, $d\gamma_k$ is a periodic orbit of ϕ_L^t , the Lagrange flow determined by the Lagrangian L . Consequently, we obtain an invariant measure μ_k whose rotation vector is ω_k . Of course, μ_k is not necessarily the minimal measure for $\beta(\omega_k)$. Nevertheless, if we choose a sub-sequence of closed curves $\{\gamma_{k_i}\}$ such that

$$\lim_{k_i \rightarrow \infty} \frac{1}{k_i} A(\gamma_{k_i}) = \liminf_{k \rightarrow \infty} \inf_{[\gamma] = \omega_k} \frac{1}{k} \int_0^k L(d\gamma(t), t) dt,$$

and if $[\gamma_{k_i}] \rightarrow \omega$, then

$$\lim_{k_i \rightarrow \infty} \frac{1}{k_i} A(\gamma_{k_i}) = \beta(\omega).$$

Clearly, there is at least one invariant measure μ such that $\mu_{k_i} \rightharpoonup \mu$, and μ is a holonomic probability measure with its rotation vector being $\omega(\mu)$. According to the definition of holonomic measure, and due to the work in [Me], we have

$$\beta(\omega) = \inf_{\nu \in \mathfrak{H}_\omega} \int \ell d\nu$$

where \mathfrak{H}_ω is a set of holonomic probability measures with the given rotation vector ω , not necessarily invariant for ϕ_L^t .

The Fenchel-Legendre transformation $\mathcal{L}_\beta: H_1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ is defined as follows

$$c \in \mathcal{L}_\beta(\rho) \iff \alpha(c) + \beta(\rho) = \langle c, \rho \rangle.$$

The concept of semi-static curves is introduced by Mather and Mañé (cf. [Ma2, Me]). A curve $\gamma: \mathbb{R} \rightarrow M$ is called c -semi-static if in time-1-periodic case we have

$$[A_c(\gamma)]|_{(t, t')} = F_c((\gamma(t), t), (\gamma(t'), t'))$$

where

$$[A_c(\gamma)]|_{(t, t')} = \int_t^{t'} \left(L(d\gamma(t), t) - \eta_c(d\gamma(t)) \right) dt + \alpha(c)(t' - t),$$

in which

$$F_c((x, t), (x', t')) = \inf_{\substack{\tau = t \bmod 1 \\ \tau' = t' \bmod 1}} h_c((x, \tau), (x', \tau')),$$

$$h_c((x, \tau), (x', \tau')) = \inf_{\substack{\xi \in C^1 \\ \xi(t) = x \\ \xi(t') = x'}} \int_\tau^{\tau'} \left(L(d\xi(t), t) - \eta_c(d\xi(t)) \right) dt + \alpha(c)(\tau' - \tau).$$

In autonomous case, the period can be considered as any positive number. Consequently, the notation of semi-static curve in this case is somehow simpler

$$[A_c(\gamma)]|_{(t, t')} = F_c(\gamma(t), \gamma(t')),$$

where

$$F_c(x, x') = \inf_{\tau > 0} h_c((x, 0), (x', \tau)).$$

Convention: Let $I \subseteq \mathbb{R}$ be an interval (either bounded or unbounded). A continuous map $\gamma: I \rightarrow M$ is called curve. If it is differentiable, the map $d\gamma: I \rightarrow TM$ is called orbit. When the implication is clear without danger of confusion, we use

the same symbol to denote the graph, $\gamma := \cup_{t \in I} (\gamma(t), t)$ is called curve and $d\gamma := \cup_{t \in I} (\gamma(t), \dot{\gamma}(t), t)$ is called orbit. In autonomous system, the terminology also applies to the image: $\gamma := \cup_{t \in I} \gamma(t)$ is called curve and $d\gamma := \cup_{t \in I} (\gamma(t), \dot{\gamma}(t))$ is called orbit.

A semi-static curve $\gamma \in C^1(\mathbb{R}, M)$ is called c -static if, in addition

$$[A_c(\gamma)|_{(t, t')}] = -F_c((\gamma(t'), \tau'), (\gamma(t), \tau))$$

holds in time-1-periodic case and

$$[A_c(\gamma)|_{(t, t')}] = -F_c(\gamma(t'), \gamma(t))$$

holds in autonomous case. An orbit $X(t) = (d\gamma(t), t \bmod 2\pi)$ is called c -static (semi-static) if γ is c -static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of c -semi-static orbits

$$\tilde{\mathcal{N}}(c) = \bigcup \{d\gamma : \gamma \text{ is } c\text{-semi static}\}$$

and call the Aubry set $\tilde{\mathcal{A}}(c)$ the union of c -static orbits

$$\tilde{\mathcal{A}}(c) = \bigcup \{d\gamma : \gamma \text{ is } c\text{-static}\}.$$

We use $\mathcal{M}(c)$, $\mathcal{A}(c)$ and $\mathcal{N}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{A}}(c)$ and $\tilde{\mathcal{N}}(c)$ from $TM \times \mathbb{T}$ to $M \times \mathbb{T}$ respectively. We have the following inclusions

$$\tilde{\mathcal{M}}(c) \subseteq \tilde{\mathcal{A}}(c) \subseteq \tilde{\mathcal{N}}(c).$$

It is showed in [Ma1, Ma2] that the inverse of the projection is Lipschitz when it is restricted to $\mathcal{A}(c)$ and $\mathcal{M}(c)$. By adding subscript s to \mathcal{N} , i.e. \mathcal{N}_s we denote its time- s -section. This principle also applies to $\tilde{\mathcal{N}}(c)$, $\tilde{\mathcal{A}}(c)$, $\tilde{\mathcal{M}}(c)$, $\mathcal{A}(c)$ and $\mathcal{M}(c)$ to denote their time- s -section respectively. For autonomous systems, these sets are defined without the time component.

On the time-1-section of Aubry set a pseudo-metric d_c is introduced by Mather in [Ma2], its definition relies on the quantity h_c^∞ . Let

$$h_c^\infty((x, s), (x', s')) = \liminf_{\substack{s=t \bmod 1 \\ t'=s' \bmod 1 \\ t'-t \rightarrow \infty}} h_c((x, t), (x', t')),$$

$$h_c^\infty(x, x') = \liminf_{k \rightarrow \infty} h_c((x, 0), (x', k)).$$

The pseudo-metric d_c on Aubry set is defined as

$$d_c((x, t), (x', t')) = h_c^\infty((x, t), (x', t')) + h_c^\infty((x', t'), (x, t)).$$

With this pseudo-metric d_c one can define equivalence class in Aubry set. The equivalence $(x, t) \sim (x', t')$ implies $d_c((x, t), (x', t')) = 0$, with which one can define quotient Aubry set $\mathcal{A}(c)/\sim$. Its element is called Aubry class, denoted by $\mathcal{A}_i(c)$, its lift to $TM \times \mathbb{T}$ is denoted by $\tilde{\mathcal{A}}_i(c)$. Thus,

$$\mathcal{A}(c) = \bigcup_{i \in \Lambda} \mathcal{A}_i(c), \quad \tilde{\mathcal{A}}(c) = \bigcup_{i \in \Lambda} \tilde{\mathcal{A}}_i(c).$$

In [?] Mather constructed an example with some quotient Aubry set homeomorphic to an interval. However, it is proved generic in [BC] that each Aubry set contains not more than $n + 1$ classes for the system with n degrees of freedom.

The definition of semi-static curve as well as of Mañé set depends on which configuration manifold under our consideration. Let $\pi : \bar{M} \rightarrow M$ be a finite covering, a curve $\gamma : \mathbb{R} \rightarrow M$ is said semi-static in \bar{M} if its lift $\bar{\gamma}$ is semi-static in \bar{M} . Accordingly,

we define $\tilde{\mathcal{N}}(c, \bar{M})$ ($\mathcal{N}(c, \bar{M})$) as the set containing all c -semi-static orbits (curves) in \bar{M} . We use the symbol $\mathcal{N}(c)$ when M is defaulted as the configuration manifold.

It is possible that $\pi\mathcal{N}(c, \bar{M}) \supsetneq \mathcal{N}(c, M)$. For instance, if $N \subset M$ is a open region such that $H_1(M, N, \mathbb{Z}) \neq H_1(M, \mathbb{Z})$, $\mathcal{N} \subset N$ and the lift of N in \bar{M} has more than one connected component, then this phenomenon takes place. But we have

Proposition 1.1. *Let $\pi : \bar{M} \rightarrow M$ be a finite covering space, then*

$$\pi\mathcal{A}(c, \bar{M}) = \mathcal{A}(c, M).$$

Proof. Pick up any $\bar{x} \in \pi\mathcal{A}(c, \bar{M})$ and any small $\delta > 0$, by definition, there exists sufficiently large $T > 0$ as well as a curve $\bar{\xi} : [0, T] \rightarrow \bar{M}$ such that $\bar{\xi}(0) = \bar{\xi}(T)$ and $[A_c(\bar{\xi})] < \delta$. Let ξ denote the project of $\bar{\xi}$ down to M , clearly, we have $[A_c(\xi)] = [A_c(\bar{\xi})] < \delta$. Let $x = \pi\bar{x}$, clearly, $x \in \mathcal{A}(c)$. \square

1.3. Outline of the proof. We use variational method to prove the result. Since the work of Mather [Ma2, Ma3], the variational method has become a powerful tool for the study of dynamical instability in positive definite Lagrange systems with multiple degrees of freedom.

In the study of diffusion in *a priori* stable systems with three degrees of freedom, mainly due to the work of Mather [Ma7], it has been widely known that the main difficulty takes place near double resonance. To describe what puzzled us and to explain the strategy of our proof, let us recall the example of Arnold and previous study on *a priori* unstable systems.

In the example of Arnold, there exists a 2-dimensional cylinder in the phase space, which is invariant and normally hyperbolic for the time-1-map determined by the Hamiltonian flow. This cylinder is foliated into a family of invariant circles, each of them has stable and unstable manifold which intersect each other transversally. Consequently, the unstable manifold of some circle intersects the stable manifold of other circles nearby, it implies the existence of a sequence of successively connected heteroclinic orbits. This structure is called transition chain by Arnold. Diffusion orbits are then constructed shadowing these heteroclinic orbits.

Such argument heavily depends on the geometric structure and it turns out to have wider range of application to use variational argument. Let us interpret the proof by variational language. Each invariant circle is the Aubry set for certain cohomology class, the stable as well as the unstable manifold is actually the graph of the differential of the weak KAM solution. Expressed as the difference of backward and forward weak KAM, the barrier function reaches its minimum at primary intersection points of these two manifolds. These homoclinic orbits and the Aubry set constitute the Mañé set in certain finite covering space. For positive definite systems, the transversal intersection implies the minimality of the homoclinic orbits as well as local heteroclinic orbits, along which the Lagrange action reaches the minimum among all those curves with the same boundary conditions. The diffusion orbits are obtained by searching for global minimizer which generates an orbit shadowing a sequence of local heteroclinic orbits.

The variational arguments still work in the case that there is no so nice geometric structure, provided the following conditions are satisfied for each cohomology class

- 1, the Aubry set is lower dimensional: $H_1(M, \mathcal{A}(c), \mathbb{Z}) \neq 0$;
- 2, the stable “manifold” intersects the unstable “manifold” transversally.

However, it turns out very difficult to verify whether the stable “manifold” intersects the unstable “manifold” transversally for each cohomology class along a path of first cohomology class. As uncountably many stable and unstable “manifolds” have to be considered, one can not get the genericity of the transversal intersection by taking the intersection of countably many open-dense sets. One possible way is to study some regularity of barrier functions with respect to some parameter. Thus, one obtains the finiteness of Hausdorff dimensions of the set of barrier functions. As such regularity is obtained in the case when a normally hyperbolic cylinder exists we are succeeded in solving the problem in *a priori* unstable case [CY1, CY2]. In fact, the regularity was obtained in [CY1] only for those barrier functions for which the minimal measure is supported on an invariant circle. It was extended in [Zo1] to all other barrier functions, which allows us to construct diffusion orbits of which the picture looks like what was constructed by Arnold, while in our previous work, the constructed orbits keep close to the cylinder when they pass through strong resonance (Birkhoff instability region). The “gap” problem was then solved.

Intuitively, diffusion orbits in *a priori* stable systems may be constructed along some resonant path. In terms of rotation vector (first homology class), each point on this curve satisfies at least one resonant condition for the system with 3 degrees of freedom. In integrable systems, each resonant path corresponds to an invariant cylinder without any hyperbolicity. Under generic perturbations, it breaks into many pieces of normally hyperbolic cylinder, but may disappear around double resonant points. It implies a bad consequence: we lost a handhold to get certain regularity of barrier functions in suitable parameter. It then becomes unclear whether there is a transition chain near double resonance.

In terms of first cohomology, strong double resonance corresponds to a convex disc with size $O(\sqrt{\epsilon})$ if the perturbation is of order $O(\epsilon)$, each piece of normally hyperbolic cylinder corresponds to a channel which extends to a small neighborhood of the disc. But it is unclear whether these channels are connected to the disc of double resonance.

The method we use to overcome this difficulty bases on following discoveries:

First, each double resonant disc is surrounded by a annulus foliated into a family of paths, along each of these paths, there is another invariant (a coordinate component of the cohomology class) except for the average action and the intersection of stable “manifold” with unstable “manifold” is nontrivial although it may not be transversal. This annulus has width of order $O(\epsilon)$.

Next, incomplete intersection of the stable and unstable “manifold” of an Aubry set does not implies that it can not be connected to any other Aubry set nearby. It does if they are c -equivalent.

Finally, the channels of normally hyperbolic cylinder reach to somewhere $\epsilon^{1+\delta}$ -close to double resonant disc ($\delta > 0$), i.e. it has overlap with the annulus. For each class in the channel, the relative homology of the Aubry set is non-trivial.

Therefore, we are able to find a path close to prescribed one, for each class on the path, the Aubry set is connected to another Aubry set nearby provided the class is also on the path. All of these connecting orbits are minimal in local sense. The diffusion orbits are constructed shadowing these successively connected orbits.

We organize the proof in following way. Section 2 is used to establish the concept of elementary weak KAM, it makes it easier to study genericity of transition chain. Section 3, 4 and 5 are devoted to study the structure of Mañé set and of Aubry set. Since these sets are symplectic invariants [Be2], we do it by studying the normal form which is put into the appendix. The truncated Hamiltonian of the normal form is a system with two degrees of freedom. In Section 3, we study the dynamics around the double resonance, and the modulus continuity of the period on energy (average action). With these preliminary works, normally hyperbolic cylinder is shown to get very close to the double resonance in Section 4, and an annulus of c -equivalence is established in Section 5. Section 6 is devoted to establish two types of local connecting orbits. The local minimality of these local connecting orbits are naturally-given, it enables us to construct global connecting orbit shadowing these local connecting orbits. It is obtained by searching for the minimizer of modified Lagrangian, which is done in Section 7. Finally in Section 8, we verify the cusp-residual property of the transition chain in nearly integrable Hamiltonian systems with three degrees of freedom. Consequently, the main result of this paper is proved.

2. ELEMENTARY WEAK KAM AND BARRIER

The concept of elementary weak KAM is introduced in this section. It plays key role in obtaining the genericity of the diffusion phenomenon.

2.1. Elementary weak KAM. The concept of c -semi-static curves can be extended to the curves only defined on \mathbb{R}^\pm , which are called forward or backward c -semi-static curves respectively. Usually one uses $\gamma_c^-(t, x, \tau): (-\infty, \tau] \rightarrow M$ to denote backward c -semi-static curve such that $\gamma_c^-(\tau) = x$, and uses $\gamma_c^+(t, x, \tau): [\tau, \infty) \rightarrow M$ to denote forward c -semi-static curve such that $\gamma_c^+(\tau) = x$. In autonomous case, one uses the notation $\gamma_c^\pm(t, x)$ such that $\gamma_c^\pm(0, x) = x$. Let

$$\begin{aligned}\tilde{\mathcal{N}}^+(c) &= \{(x, \dot{x}, \tau) \in TM \times \mathbb{T} : \pi_x \phi_L^t(z, \tau)|_{[\tau, +\infty)} \text{ is } c\text{-semi-static}\}, \\ \tilde{\mathcal{N}}^-(c) &= \{(x, \dot{x}, \tau) \in TM \times \mathbb{T} : \pi_x \phi_L^t(z, \tau)|_{(-\infty, \tau]} \text{ is } c\text{-semi-static}\},\end{aligned}$$

where $0 \leq \tau < 1$, $\phi_L^t(x, \dot{x}, \tau)$ denotes the orbit of the Lagrangian flow with the initial value (x, \dot{x}) at the time τ , π_x denotes the standard projection along tangent fiber $\pi_x(x, \dot{x}) = x$. The corresponding orbits are called forward (backward) semi-static orbit set respectively. These two sets are upper semi-continuous for the cohomology class.

Proposition 2.1. (see [Bo]) *If the Lagrangian L is of Tonelli type, for each point $(x, \tau) \in M \times \mathbb{T}$, there is at least one $\gamma_c^\pm(t, (x, \tau))$ which is forward (backward) semi-static curve.*

As both the ω -limit set of $d\gamma_c^+$ and the α -limit set of $d\gamma_c^-$ are in the Aubry set one can define

$$W_c^\pm = \bigcup_{(x,\tau) \in M \times \mathbb{T}} \left\{ (x, \tau), \frac{d\gamma_c^\pm(\tau, (x, \tau))}{dt} \right\},$$

and call W_c^+ and W_c^- the stable and unstable set of the c -minimal measure respectively. If for some $(x, \tau) \in M \times \mathbb{T}$ we have $d\gamma_c^-(\tau) = d\gamma_c^+(\tau)$, then there is an orbit passing through the point $((x, \tau), d\gamma_c^-(\tau, (x, \tau)))$ which is either in some Aubry set or homoclinic to this Aubry set.

When the Aubry set contains only one class, the stable as well as the unstable set has its own generating function u_c^\pm such that $W_c^\pm = \text{Graph}(du_c^\pm)$ holds almost everywhere. These functions are weak KAM solutions, which are the fixed points of so called Lax-Oleinik operator [Fa1, Fa2]. We use u_c^\pm to denote the weak KAM solution for the Lagrangian $L - \eta_c$, where η_c is a closed form with $[\eta_c] = c$. These functions are Lipschitz, thus differentiable almost everywhere. At each differentiable point (x, τ) , $(x, \tau, \partial_x u^-(x, \tau))$ uniquely determines backward c -semi static curve $\gamma_x^-: (-\infty, \tau] \rightarrow M$ such that $\gamma_x^-(\tau) = x$, $\dot{\gamma}_x^-(\tau) = \partial_y H(x, \tau, \partial_x u^-(x, \tau))$. Similarly, $(x, \tau, \partial_x u^+(x, \tau))$ uniquely determines forward c -semi static curve $\gamma_x^+: [\tau, \infty) \rightarrow M$ such that $\gamma_x^+(\tau) = x$, $\dot{\gamma}_x^+(\tau) = \partial_y H(x, \tau, \partial_x u^+(x, \tau))$.

Given a cohomology class c , we use $\{\tilde{\mathcal{A}}_c^i\}_{i \in \Lambda} \subset TM$ to denote the set of an Aubry class, use $\{\mathcal{A}_c^i\}_{i \in \Lambda} \subset M$ to denote the projected set along the tangent fibers. Here Λ is an index set so that $\tilde{\mathcal{A}}(c) = \bigcup_{i \in \Lambda} \tilde{\mathcal{A}}_c^i$. We also use the symbol $\tilde{\mathcal{M}}_c^i = \text{supp} \mu_i$ where μ_i is an ergodic component of the c -minimal measure μ_c , and let $\mathcal{M}_c^i = \pi \tilde{\mathcal{M}}_c^i$.

Proposition 2.2. ([Fa2]) *Let u_c^\pm and $u_c'^\pm$ be two weak-KAM solutions for c . Restricted on each Aubry class their difference keeps constant: $(u_c^\pm - u_c'^\pm)|_{\mathcal{A}_c^i} = \text{constant}$.*

Recall the definition of h_c^∞ in the introduction. We use the symbol h_L^∞ to denote the quantity defined in the same way for L with $c = 0$, and drop the subscript L when it is clearly defined. Let $z = (x, \tau), z' = (x', \tau')$. By definition, there is a subsequence $\{k_i \in \mathbb{Z}\}$ with $k_i \rightarrow \infty$ such that

$$\lim_{k_i \rightarrow \infty} h^{k_i}(z, z') = h^\infty(z, z').$$

Let $\gamma_i: [\tau, k_i + \tau'] \rightarrow M$ be the curve which joins x with x' and realizes the quantity $h^{k_i}(z, z')$. Given any large number N , the set $\{\gamma_i|_{[\tau, N]}\}_{i \geq i_0}$ is C^1 -compact. Thus, there exists at least one curve $\gamma: [\tau, \infty) \rightarrow M$ such that, restricted on $[\tau, N]$, it is an accumulation point of $\{\gamma_i|_{[\tau, N]}\}_{i \geq i_0}$. Obviously, this curve is forward semi-static.

The quantity $h_c^\infty(z, z')$ is a weak-KAM solution if we consider it as the function of z or of z' . Let us consider the case that the c -minimal invariant measure has finitely many ergodic components. In this case, this function has some kind of continuity.

Theorem 2.1. *Let $\{L_\xi\}$ be a sequence of Lagrangian, converging to L in C^2 -topology as $\xi \rightarrow 0$ when they are restricted on any bounded regions of $TM \times \mathbb{T}$. We assume that the minimal measure for L consists of m ergodic components $\mu^1, \mu^2, \dots, \mu^m$, and the distance from $(x, \tau) \in \mathcal{M}^i$ to the Aubry set for L_ξ , $d((x, \tau), \mathcal{A}_{L_\xi}) \rightarrow 0$ as $\xi \rightarrow 0$. Then*

$$\lim_{\xi \rightarrow 0} h_{L_\xi}^\infty((x, \tau), (x', \tau')) = h^\infty((x, \tau), (x', \tau')).$$

Proof. We only need to prove it on the time-1-section, i.e. for $\tau = \tau' = 0$, so, we omit the symbol for the component of τ . For each $\epsilon > 0$, there exists $k > 0$ such that $|h^\infty(x, x') - h^k(x, x')| < \epsilon$. Let α and α_ξ denote the minimal average action of L and L_ξ respectively. Let $\gamma^k: [0, k] \rightarrow M$ be the curve such that $\gamma^k(0) = x$, $\gamma^k(k) = x'$ and

$$[A(\gamma^k)] = \int_0^k L(d\gamma^k(t), t) dt + k\alpha = h^k(x, x')$$

For any $k' \geq k$ we construct an absolutely continuous curve $\zeta: [0, k'] \rightarrow M$ such that $\zeta(t - k' + k) = \gamma^k(t)$ for $t \in [k' - k, k']$ and $[A_{L_\xi}(\zeta|_{[0, k' - k]})] = \int_0^{k' - k} L_\xi(d\zeta(t), t) dt + (k' - k)\alpha_\xi = h_\xi^{k' - k}(x, x)$. Thus,

$$\begin{aligned} [A_{L_\xi}(\zeta)] &\leq h_{L_\xi}^{k' - k}(x, x) + h^k(x, x') + k|\alpha - \alpha_\xi| \\ &\quad + \left| \int_0^k (L - L_\xi)(d\gamma^k(t), t) dt \right|. \end{aligned}$$

Since $d(x, \mathcal{A}_{L_\xi}|_{t=0}) \rightarrow 0$ as $\xi \rightarrow 0$ we see that $\liminf_{k' \rightarrow k} h_\xi^{k' - k}(x, x) \rightarrow 0$ as $\xi \rightarrow 0$. Since α is continuous in the Lagrangian and ϵ is arbitrarily small we see that

$$\limsup_{\xi \rightarrow 0} h_{L_\xi}^\infty(x, x') \leq h^\infty(x, x').$$

Therefore, to complete the proof, we only need to show

$$(2.1) \quad \liminf_{\xi \rightarrow 0} h_{L_\xi}^\infty(x, x') \geq h^\infty(x, x').$$

Let $\gamma_\xi^{k_\ell}: [0, k_\ell] \rightarrow M$ be a curve such that $\gamma_\xi^{k_\ell}(0) = x$, $\gamma_\xi^{k_\ell}(k_\ell) = x'$ and

$$[A_\xi(\gamma_\xi^{k_\ell})] = h_{L_\xi}^{k_\ell}(x, x') \rightarrow h_{L_\xi}^\infty(x, x'),$$

where $k_\ell \rightarrow \infty$ is a sequence of integers. Let $O_\epsilon(S)$ denote the ϵ -neighborhood of the set S . For small $\epsilon > 0$, there exist some j with $1 \leq j \leq m$, an integer $k_j \in [0, k_\ell]$ and $x_j \in \mathcal{M}_0^j = \mathcal{M}^j|_{t=0}$ such that $\gamma_\xi^{k_\ell}(k_j) \in O_\epsilon(x_j)$ provided k_ℓ is sufficiently large.

Let us consider those ergodic components of minimal measure for L of which the support is approached by $d\gamma_\xi^{k_\ell}$ as $\xi \rightarrow 0$: $\{d\gamma_\xi^{k_\ell}\} \cap O_\epsilon(\tilde{\mathcal{M}}^j) \neq \emptyset$. We number some j as j_1 if some $x_1 \in \mathcal{M}_0^{j_1}$ exists such that $\gamma_\xi^{k_\ell}(k_1) \in O_\epsilon(x_1)$ and for each $k < k_1$, $\gamma_\xi^{k_\ell}(k)$ does not fall into ϵ -neighborhood of any \mathcal{M}_0^j . Let $k'_1 \geq k_1$ be the integer such that $\gamma_\xi^{k_\ell}(k'_1) \in O_\epsilon(x_1)$ and $\gamma_\xi^{k_\ell}(k) \notin O_\epsilon(x_1)$ for all $k > k'_1$. We number some $j_2 \neq j_1$ if some $k_2 > k'_1$ and some $x_2 \in \mathcal{M}_0^{j_2}$ exist such that $\gamma_\xi^{k_\ell}(k_2) \in O_\epsilon(x_2)$, let $k'_2 \geq k_2$ be the integer such that $\gamma_\xi^{k_\ell}(k'_2) \in O_\epsilon(x_2)$ and $\gamma_\xi^{k_\ell}(k') \notin O_\epsilon(x_2)$ for all $k > k'_2$. Inductively, one obtains $x_i \in \mathcal{M}_0^{j_i}$ ($i = 1, 2, \dots, m' \leq m$) and

$$0 \leq k_1 \leq k'_1 \leq \dots \leq k_{m'} \leq k'_{m'} \leq k_\ell.$$

Obviously, there exist small $\delta = \delta(\epsilon) > 0$ and large integer $K = K(\epsilon)$ such that

$$|k_j - k'_{j-1}| \leq K, \quad \forall k_\ell \rightarrow \infty$$

provided $|\xi| < \delta$. Otherwise, there would exist also an ergodic component ν of the minimal invariant measure such that $\nu \neq \mu^j$ for all $0 \leq j \leq m$, but it is absurd.

Given small $\epsilon > 0$, let k_ℓ be the integer such that $|h_{L_\xi}^{k_\ell}(x, x') - h_{L_\xi}^\infty(x, x')| < \epsilon$. Let $\bar{x}_j = \gamma_\xi^{k_\ell}(k_j)$, $\tilde{x}_j = \gamma_\xi^{k_\ell}(k'_j)$, we choose an absolutely continuous curve $\zeta_j: [0, k'_j] \rightarrow M$

such that $\zeta_j(0) = \bar{x}_j$, $\zeta_j(k_\ell^j) = \tilde{x}_j$ and $[A(\zeta_j)] = h^{k_\ell^j}(\bar{x}_j, \tilde{x}_j)$. As $\bar{x}_j, \tilde{x}_j \in O_\epsilon(x_j)$ we can choose sufficiently large k_ℓ^j such that

$$|h^{k_\ell^j}(\bar{x}_j, \tilde{x}_j)| < C\epsilon,$$

where $C = C(L)$ is a constant depending on L only. As $\|\bar{x}_j - \tilde{x}_j\| \leq 2\epsilon$, for any integer i we have

$$h_{L_\xi}^i(\bar{x}_j, \tilde{x}_j) > -C\epsilon.$$

For any large integer $k \in \mathbb{Z}$, we can construct an absolutely continuous curve $\zeta: [0, k'] \rightarrow M$ ($k \leq k'$) joining x with x' such that

$$\zeta(t) = \begin{cases} \gamma_\xi^{k_\ell}(t - \tau_{j-1}), & \text{if } k'_{j-1} + \tau_{j-1} \leq t \leq k_j + \tau_{j-1}, \\ \zeta_j(t - k_j - \tau_j), & \text{if } k_j + \tau_{j-1} \leq t \leq k_j + \tau_j \end{cases}$$

where $\tau_j = \sum_{j=1}^j (k_\ell^j - k'_j + k_j)$, $k' = k_\ell + \tau_{m'}$. The action of L along this curve is easily estimated

$$\begin{aligned} h^{k'}(x, x') - h_{L_\xi}^k(x, x') &\leq [A(\zeta)] - h_{L_\xi}^k(x, x') \\ &\leq 2m'(C\epsilon + K|\alpha - \alpha_\xi|) \\ &\quad + \sum_{j=1}^{m'} \left| \int_{k'_{j-1}}^{k_j} (L - L_\xi)(d\gamma_\xi^\ell(t), t) dt \right|. \end{aligned}$$

As $|k_j - k_{j-1}| \leq K$, and K is independent of ξ when ξ is sufficiently close to 0, we see that the inequality (2.1) holds. This completes the proof. \square

Corollary 2.1. *Let $c_i \rightarrow c$ be a sequence of cohomology classes. We assume that the c -minimal measure consists of m ergodic components $\mu_c^1, \mu_c^2, \dots, \mu_c^m$, $(x, \tau) \in \mathcal{M}_c^j$ and $d((x, \tau), \mathcal{M}(c_i)) \rightarrow 0$ for some $0 \leq j \leq m$, as $c_i \rightarrow c$. Then*

$$\lim_{c_i \rightarrow c} h_{c_i}^\infty((x, \tau), (x', \tau')) = h_c^\infty((x, \tau), (x', \tau')).$$

From the proof one can see that the function h_c^∞ is lower semi-continuous in c if the c -minimal measure is assumed to have finitely many ergodic components:

$$\liminf_{c' \rightarrow c} h_{c'}^\infty(z, z') \geq h_c^\infty(z, z').$$

If the c -minimal measure has finitely many ergodic components, we can introduce the concept of *elementary* weak KAM solution. One can choose finitely many non-negative functions $g_i: M \times \mathbb{T} \rightarrow \mathbb{R}$ such that its support has no intersection with a small neighborhood of \mathcal{M}_c^i and the minimal measure for the Lagrangian $L_{c,i,\epsilon} = L_c + \epsilon g_i$ is uniquely supported on \mathcal{M}_c^i . By the theory of weak KAM ([Fa2]), there is exactly one pair of weak KAM solutions denoted by $u_{c,i,\epsilon}^\pm$ and

$$h_{L_{c,i,\epsilon}}^\infty(z, z') = u_{c,i,\epsilon}^-(z') - u_{c,i,\epsilon}^+(z).$$

Choosing $z \in \mathcal{M}_c^i$, in virtue of Theorem 2.1, one has $h_{L_{c,i,\epsilon}}^\infty(z, z') \rightarrow h_c^\infty(z, z')$ as $\epsilon \rightarrow 0$. Since $g_i = 0$ in the neighborhood of \mathcal{M}_c^i , $u_{c,i,\epsilon}^+(z)$ remains unchanged as $\epsilon \rightarrow 0$. Thus, there is a Lipschitz function $u_{c,i}^-$ such that $u_{c,i,\epsilon}^- \rightarrow u_{c,i}^-$ as $\epsilon \rightarrow 0$. Clearly this $u_{c,i}^-$ is a weak-KAM solution for L_c . Similarly, we can see that $u_{c,i,\epsilon}^+ \rightarrow u_{c,i}^+$ as $\epsilon \rightarrow 0$.

Definition 2.1. (ELEMENTARY WEAK-KAM SOLUTION). Assume that the minimal measure for L_c consists of finitely many ergodic components $\mu_c^1, \mu_c^2, \dots, \mu_c^m$. A weak KAM $u_{c,i}^\pm$ of L_c is called elementary if $u_{c,i}^\pm = \lim_{\epsilon \rightarrow 0} u_{c,i,\epsilon}^\pm$ where $u_{c,i,\epsilon}^\pm$ is the weak KAM solutions of $L_{c,i,\epsilon}$, of which the minimal measure is uniquely ergodic and $L_{c,i,\epsilon} \rightarrow L_c$ as $\epsilon \rightarrow 0$.

It is not necessary that $u_{c,i}^\pm$ is a pair of conjugate weak KAM. Clearly, if $(x, t) \in \mathcal{M}_c^i$

$$\begin{aligned} h_c^\infty((x, t), (x', t')) &= u_{c,i}^-(x', t') - u_{c,i}^+(x, t), \\ h_c^\infty((x', t'), (x, t)) &= u_{c,i}^+(x, t) - u_{c,i}^-(x', t'). \end{aligned}$$

These elementary weak KAM solutions generate all weak KAM solutions in the following sense.

Proposition 2.3. Assume the minimal measure consists of m ergodic components. For each weak KAM solution u^\pm , there exist m' ($m' \leq m$) constants $d_1^\pm, \dots, d_{m'}^\pm$, and m' open domains $D_1^\pm, \dots, D_{m'}^\pm$ such that they do not overlap each other, $M = \cup_{1 \leq i \leq m'} \bar{D}_i^\pm$ and

$$(2.2) \quad u^\pm|_{D_i^\pm} = u_i^\pm + d_i^\pm, \quad \forall 1 \leq i \leq m'.$$

Proof. It is deduced from the Lipschitz property of u^- that it is differentiable almost every where. Let x be a point where u^- is differentiable, $du^-(x)$ determines a unique backward semi static orbit $d\gamma_c^i: (-\infty, 0] \rightarrow M$ whose α -limit set is in certain Aubry set $\tilde{\mathcal{A}}_c^i$. By definition we have

$$u^-(x) - u^-(\gamma_c^i(-t)) = \int_{-t}^0 L_c(d\gamma_c^i(s), s) ds + \alpha(c)t$$

let $t_k \rightarrow \infty$ such that $\gamma_c^i(-t_k) \rightarrow x' \in \mathcal{A}_c^i$, it follows from Proposition (2.2) that

$$(2.3) \quad u^-(x) = h_c^\infty(x', x) + u^-(x') = u_{c,i}^-(x) + d_i.$$

If $x^* \in M$ is another point where $du^-(x^*)$ determines a backward semi-static orbits whose α -limit set is also contained in $\tilde{\mathcal{A}}_c^i$, we then obtain (2.3) for $u^-(x^*)$ with the same d_i . All these points constitute a set connected with \mathcal{A}_c^i , there are not more than m connected sets such that (2.2) holds. \square

Theorem 2.2. Let $c_i \rightarrow c$ be a sequence of cohomology and assume that the minimal measure consists of finite ergodic components for each c_i and c . Let $\mathcal{M}_{c_i}^j, \mathcal{M}_c^j$ be the support for the ergodic minimal measure $\mu_{c_i}^j$ and μ_c^j respectively, let $u_{c_i}^-$ and u_c^- be the corresponding elementary weak KAM solution. If $\mu_{c_i}^j \rightarrow \mu_c^j$ as $c_i \rightarrow c$, then $u_{c_i}^- \rightarrow u_c^-$ in C^0 -topology.

Proof. It follows from the continuity of $h_c^\infty(x, x')$ in c shown in Theorem 2.1 and the definition of the elementary weak KAM solution. \square

In terms of conjugate pair of weak KAM solution, one has a definition of Mañé set in [Fa2]. For the purpose of this paper, we would like to use elementary weak KAM solution. Recall the definition of the barrier function in [Ma2]:

$$B_c^*(x) = \min_{\xi, \zeta \in \mathcal{M}_0(c)} \{h_c^\infty(\xi, x) - h_c^\infty(x, \zeta) + h_c^\infty(\xi, \zeta)\}.$$

If the minimal measure consists of finitely many ergodic components, we introduce barrier functions in terms of elementary weak KAM solutions: given $z = (x, \tau) \in M \times \mathbb{T}$, we set

$$(2.4) \quad B_{c,i,j}(z) = u_{c,i}^-(z) - u_{c,j}^+(z).$$

which measures the minimum of the action along all those curves passing through z and joining \mathcal{M}_c^i to \mathcal{M}_c^j . For autonomous systems, this barrier function is independent of time. Obviously, each semi static curve corresponds to a minimum of $u_{i,c}^- - u_{j,c}^+$ if its α -limit set intersects \mathcal{M}_c^i and its ω -limit set intersects \mathcal{M}_c^j .

2.2. Minimal Homoclinic Orbits to Aubry Set. To extend the concept of elementary weak KAM solution to universal covering space, let us reveal some properties of minimal homoclinic orbit to Aubry set.

Given a curve $\gamma: \mathbb{R} \rightarrow M$, we call $d\gamma = (\gamma, \dot{\gamma})$ a homoclinic orbit to some Aubry set $\tilde{\mathcal{A}}$ if it does not stay in the Aubry set, but its ω -limit set as well as the α -limit set is contained in the Aubry set:

$$\alpha(d\gamma) \subseteq \tilde{\mathcal{A}} \quad \text{and} \quad \omega(d\gamma) \subseteq \tilde{\mathcal{A}}.$$

Correspondingly, we call γ homoclinic curve. The existence of homoclinic orbits to Aubry sets has been studied in a few papers, see [Bo, Be1, Cui, Zhe, Zo2].

The existence of such homoclinic orbits is closely related to the issue whether the Čech homology group $H_1(M, \mathcal{A}, \mathbb{R})$ is non-trivial (we need to consider $H_1(M \times \mathbb{T}, \mathcal{A}, \mathbb{R})$ for time-periodically dependent Lagrangian). It is defined as the inverse limit $\lim_{A \subset U} H_1(M, U, \mathbb{R})$, where U is an open neighborhood of \mathcal{A} . There exists a small open neighborhood U_0 of \mathcal{A} such that $\text{rank} H_1(M, U, \mathbb{R}) = \text{rank} H_1(M, \mathcal{A}, \mathbb{R})$ provided $U \subseteq U_0$.

Let \bar{M} be a covering of M such that $\pi_1(\bar{M}) = \ker(\mathfrak{H}: \pi_1(M) \rightarrow H_1(M, \mathbb{R}))$ where \mathfrak{H} denotes the Hurewicz homomorphism. The group of Deck transformation of this covering space is

$$H = \text{im}(\mathfrak{H}: \pi_1(M) \rightarrow H_1(M, \mathbb{R})).$$

Let U be an open neighborhood of \mathcal{A} such that $\text{rank} H_1(M, U, \mathbb{Z}) = \text{rank} H_1(M, \mathcal{A}, \mathbb{R})$. Let $K = i_* H_1(U, \mathbb{Z}) \subset H$ and $G = H/K$, then G is a free Abel group. To each orbit $(\gamma, \dot{\gamma}): \mathbb{R} \rightarrow M$ homoclinic to $\tilde{\mathcal{A}}$, an element $[\gamma] \in G$ is associated.

If the group G is non-trivial, there is a flat \mathbb{F} of the α -function containing the cohomology class. A set $\mathbb{F} \subset H^1(M, \mathbb{R})$ is called flat if the function α is affine when it is restricted on \mathbb{F} , not affine for any set properly contains \mathbb{F} . The dimension of this flat is not smaller than $r = \text{rank} H_1(M, \mathcal{A}, \mathbb{Z})$ and the Aubry set is the same for all classes in the interior of \mathbb{F} (see [Ms]).

In this paper, we are interested in so-called *minimal* homoclinic orbits. Let \check{M} be a covering manifold of M such that $\pi_1(\check{M}) = \pi_1(U)$. A curve $\gamma: \mathbb{R} \rightarrow M$ is called \check{M} semi-static if the lift of γ to \check{M} , $\tilde{\gamma}: \mathbb{R} \rightarrow \check{M}$ is semi-static. A homoclinic orbit $d\gamma$ is called *minimal* if the lift $\tilde{\gamma}: \mathbb{R} \rightarrow \check{M}$ is semi-static.

Theorem 2.3. *If there is only one Aubry class and $\text{rank} H_1(M, \mathcal{A}, \mathbb{R}) = r > 0$, then there are at least $r + 1$ minimal homoclinic orbits. If $\mathcal{M}(c) \supsetneq \mathcal{M}(c')$ for $c \in \partial \mathbb{F}$ and $c' \in \text{int} \mathbb{F}$, then there are infinitely many c -minimal homoclinic orbits.*

The existence of at least $r + 1$ homoclinic orbits is proved in [Be1], these orbits are minimal. Let us briefly describe how to find these $r + 1$ minimal homoclinic orbits. Given a point $x \in \mathcal{A}$ and $g \in G$, we denote by $\xi_i: [-i, i] \rightarrow M$ the minimizer of

$$h_g^i(x) = \inf_{\substack{\xi_i(-i)=\xi_i(i)=x \\ [\xi_i]=g}} \int_{-i}^i L(\xi_i(s), \dot{\xi}_i(s)) ds + 2i\alpha$$

Obviously, $\{\|\dot{\xi}_i(t)\| : t \in [-i, i]\}$ is uniformly bounded for $i > 0$. Consequently, $\{\|\ddot{\xi}_i(t)\| : t \in [-i, i]\}$ is also uniformly bounded for each i , because of positive definiteness of L . Let

$$h_g^\infty(x) = \liminf_{i \rightarrow \infty} h_g^i(x),$$

there exists a subsequence of i_j such that $h_g^{i_j}(x) \rightarrow h_g^\infty(x)$. Obviously, h_g^∞ keeps constant on each Aubry class. By diagonal extraction argument we can find a subsequence of ξ_{i_j} which converges C^1 -uniformly on each compact interval to a C^1 -curve $\gamma: \mathbb{R} \rightarrow M$. In this sense, $\gamma: \mathbb{R} \rightarrow M$ is called an accumulation point of $\{\xi_{i_j}\}$. Clearly, each accumulation point is \tilde{M} semi-static and there is at least one accumulation point γ_1 with non-zero homology $[\gamma_1] \neq 0$.

Clearly, some $a > 0$ exists such that $h_g^\infty \geq a$ holds for each $g \in G$ and $h_g \rightarrow \infty$ as $|g| \rightarrow \infty$. Thus, for each $g \in G$, there are finitely many accumulation points of $\{\xi_{i_j}\}$ with non-zero homology, denoted by $\gamma_1, \dots, \gamma_i$. Clearly $\sum_{j=1}^i [\gamma_j] = g$. As G is r -dimensional, at least $r + 1$ geometrically different minimal homoclinic orbits exist.

Let us look at these homoclinic orbits from another point of view. For certain finite covering manifold, the lift of the Aubry set has several connected components (several Aubry classes). By a result in [CP], these Aubry classes are connected by semi-static orbits. The projection of these semi-static orbits are nothing else but minimal homoclinic orbits. For a finite covering manifold $\tilde{\pi}: \tilde{M} \rightarrow M$, the fiber $\tilde{\pi}^{-1}x$ contains finitely many points. For a simple curve $\phi: [0, 1] \rightarrow M$ such that $\phi(0) = \phi(1) = x$, there is a lift of ϕ such that $\bar{\phi}(0) = \bar{x}_0 \in \tilde{\pi}^{-1}x$. By monodromy theorem, $\bar{\phi}(1) \in \tilde{\pi}^{-1}x$ is uniquely determined by its class $[\phi] \in \pi_1(M)$. Let g_1, g_2, \dots, g_r be the generators of G , $\phi_1, \phi_2, \dots, \phi_r$ be closed path so that $[\phi_i] = g_i$, $\phi_i(0) = x$ for $i = 1, 2, \dots, r$. If \tilde{M} is chosen so that $\bar{\phi}_i(0) = \bar{x}_0$ and $\bar{\phi}_i(1) \neq \bar{\phi}_j(1)$, there will be at least $2r$ Aubry classes for this covering manifold. Among the semi-static orbits connecting different Aubry classes for the covering manifold, there are at least $r + 1$ orbits whose projection is different from each other.

Let $G_m \subset G$ be defined such that $g \in G_m$ if and only if \exists minimal homoclinic orbit $d\gamma$ such that $[\gamma] = g$. We say that there are k -types of minimal homoclinic orbits if G_m contains exactly k elements.

Theorem 2.4. *If $M = \mathbb{T}^n$, $H_1(M, \mathcal{A}, \mathbb{Z}) \neq 0$ and \mathcal{A} contains a set homeomorphic to \mathbb{T}^{n-1} , there are two types of minimal homoclinic orbits only.*

Proof. In this situation one has $G = \mathbb{Z}$. Each standard generator $e_i \in H_1(\mathbb{T}^n, \mathbb{Z})$ with $i > 1$ can be represented by a closed curve in \mathcal{A} . Let $g = ke_1$ with $k > 1$. If there is a minimal homoclinic orbits $(\gamma, \dot{\gamma})$ such that $[\gamma] = g$, there must be some points $x = \gamma(t_0) \in \mathcal{A}$ but $(\gamma(t_0), \dot{\gamma}(t_0)) \notin \tilde{\mathcal{A}}$.

As $x \in \mathcal{A}$, there is a unique vector v such that $(x, v) \in \tilde{\mathcal{A}}$. Given any $\epsilon > 0$, there is static curve $\xi: \mathbb{R} \rightarrow M$ and $s_1 < s_2$ such that $\xi(s_0), \xi(s_1)$ are in ϵ -neighborhood

of x , $\|\dot{\xi}(s_0) - v\| < \epsilon$, $\|\dot{\xi}(s_1) - v\| < \epsilon$, and the action along this curve is also small $[A(\xi)]_{[s_0, s_1]} < \epsilon$.

Let $\tau_1^- = t_0 - t^- > 0$, $\tau_2^+ = t^+ - t_0 > 0$, $\tau_1^+ = s_0^+ - s_0 > 0$ and $\tau_2^- = s_1 - s_1^- > 0$ be suitably small numbers. We join $\gamma(t^-)$ to $\xi(s_0^+)$ by the curve $\zeta_1: [-\tau_1^-, \tau_1^+] \rightarrow M$ which minimizes the action

$$[A(\zeta_1)]_{[-\tau_1^-, \tau_1^+]} = \inf_{\substack{\zeta(-\tau_1^-) = \gamma(t^-) \\ \zeta(\tau_1^+) = \xi(s_0^+)}} \int_{-\tau_1^-}^{\tau_1^+} L(\xi(s), \dot{\xi}(s)) ds + (\tau_1^+ + \tau_1^-)\alpha,$$

and join $\xi(s_1^-)$ to $\gamma(t^+)$ by the curve $\zeta_2: [-\tau_2^-, \tau_2^+] \rightarrow M$ which minimizes the action

$$[A(\zeta_2)]_{[-\tau_2^-, \tau_2^+]} = \inf_{\substack{\zeta(-\tau_2^-) = \xi(s_1^-) \\ \zeta(\tau_2^+) = \gamma(t^+)}} \int_{-\tau_2^-}^{\tau_2^+} L(\xi(s), \dot{\xi}(s)) ds + (\tau_2^+ + \tau_2^-)\alpha.$$

We define a continuous curve $\gamma': \mathbb{R} \rightarrow M$ by

$$\gamma'(t) = \begin{cases} \gamma(t), & t \in (-\infty, t^-], \\ \zeta_1(t - \Delta_1), & t - \Delta_1 \in [-\tau_1^-, \tau_1^+], \\ \xi(t - \Delta_2), & t - \Delta_2 \in [s_0^+, s_1^-], \\ \zeta_2(t - \Delta_3), & t - \Delta_3 \in [-\tau_2^-, \tau_2^+], \\ \gamma(t - \Delta_4), & t - \Delta_4 \in [t^+, \infty), \end{cases}$$

where $\Delta_1 = t^- + \tau_1^-$, $\Delta_2 = t^- + \tau_1^- + \tau_1^+ - s_0^+$, $\Delta_3 = t^- + \tau_1^- + \tau_1^+ - s_0^+ + s_1^- + \tau_2^-$ and $\Delta_4 = t^- + \tau_1^- + \tau_1^+ - s_0^+ + s_1^- + \tau_2^- + \tau_2^+ - t^+$. By exploiting the *curve shorten* lemma in Riemannian geometry as did in [Ma2] we find that

$$[A(\gamma)]_{[t^-, t^+]} + [A(\xi)]_{[s_0, s_0^+] \cup [s_1^-, s_1]} > [A(\zeta_1)]_{[-\tau_1^-, \tau_1^+]} + [A(\zeta_2)]_{[-\tau_2^-, \tau_2^+]}$$

if $\xi(s_0) = \xi(s_1) = x$ and $\dot{\xi}(s_0) = \dot{\xi}(s_1) = v \neq \dot{\gamma}(t_0)$. As $x \in \mathcal{A}$, $(\xi(s_0), \dot{\xi}(s_0))$, $(\xi(s_1), \dot{\xi}(s_1))$ can be arbitrarily close to (x, v) by choosing suitable s_0 and s_1 , this inequality still hold in our case. Note that the quantity $[A(\xi)]_{[s_0, s_1]}$ can be arbitrarily close to zero, we see that

$$[A(\gamma)]_{[t_{-1}, t_1]} > [A(\gamma')]_{[t_{-1}, t_1 + \Delta_4]}$$

if $t_{-1} < t^-$ and $t_1 > t^+$. Clearly, we have $[\gamma'] = [\gamma]$. This contradicts the fact that γ is minimal. On the other hand, from Theorem 2.3, we obtain the existence of 2 minimal homoclinic orbits. This completes the proof. \square

In general case, G_m may contains infinitely many elements. Whether G_m is finite is closely related to the sequence of action along these minimal homoclinic orbits. For each $g \in G$, we define

$$h_g^k(x, x) = \inf_{\substack{\xi(0) = \xi(k) = x \\ \xi \in C^1, [\xi] = g}} \int_0^k L(\xi(s), \dot{\xi}(s)) ds + k\alpha.$$

$$h_g^\infty(x, x) = \liminf_{k \rightarrow \infty} h_g^k(x, x).$$

It is easy to see that $h_g^\infty(x, x) \rightarrow \infty$ as $\|g\| \rightarrow \infty$, which follows from the fact $H_1(\mathbb{T}^n, \mathcal{A}, \mathbb{Z}) \neq 0$ that $h_g^\infty(x, x) > 0$ for any $g \neq 0$. If $h_g^\infty(x, x)$ remains bounded as $\|g\| \rightarrow \infty$, there would be a minimal measure whose support is obviously not contained in $\tilde{\mathcal{A}}$, but it is absurd.

If the Aubry set contains only one class, as a function of x , $h_g^\infty(x, x)$ keeps constant on the Aubry set. So it makes sense let $h_g^\infty = h_g^\infty(x, x)$ for $x \in \mathcal{A}$. Obviously,

$$h_{g_1+g_2}^\infty \leq h_{g_1}^\infty + h_{g_2}^\infty.$$

Proposition 2.4. *If there is a infinite sequence $\{g_i\} \subset G$ such that*

$$h_{g_i+g_j}^\infty < h_{g_i}^\infty + h_{g_j}^\infty,$$

then G_m contains infinitely many elements.

The definition of $h_g^k(x, x)$ can be extended $h_g^k(x, x')$ for $x \neq x'$. Recall the covering space $\tilde{\pi}: \tilde{M} \rightarrow M = \mathbb{T}^n$ such that $\pi_1(\tilde{M}) = \pi_1(U)$, where U is a open neighborhood of $\mathcal{A} \subset \mathbb{T}^n$ such that $H_1(M, U, \mathbb{R}) = H_1(M, \mathcal{A}, \mathbb{R})$. Let $D = \{\bar{x} : \bar{x}_i \in [0, 1]\} \subset \mathbb{R}^n$ be the fundamental domain for \mathbb{T}^n and use the same symbol to denote its projection to \tilde{M} as well. For each closed path $\phi: [0, 1] \rightarrow M$, there is a lift $\check{\phi}: [0, 1] \rightarrow \tilde{M}$ with $\check{\phi}(0) \in D$. Note that the fundamental group of \mathbb{T}^n is commutative. Because of the monodromy theorem and $\pi_1(U) = \pi_1(M)$, $\check{\phi}(1) \in \tilde{M}$ is uniquely determined by the homological type $[\phi] \in H_1(\mathbb{T}^n, U, \mathbb{Z})$.

For a curve $\xi: [0, k] \rightarrow M$ with $\xi(0) = x$, $\xi(k) = x'$, we denote by $\check{\xi}$ the lift of ξ such that $\check{\xi}(0) \in D$. We say $[\xi] = g$ if $[\phi] = g$ holds for any closed curve ϕ such that the lift to \tilde{M} satisfies the condition that $\check{\phi}(0) \in D$ and $\check{\phi}(1) = \check{\xi}(k)$. Thus the following is well-defined:

$$h_g^k(x, x') = \inf_{\substack{\xi(0)=x \\ \xi(k)=x' \\ [\xi]=g \\ \xi \in C^1}} \int_0^k L(\xi(s), \dot{\xi}(s)) ds + k\alpha,$$

$$h_g^\infty(x, x') = \liminf_{k \rightarrow \infty} h_g^k(x, x').$$

Clearly, $h_g^\infty(x, x') \rightarrow \infty$ as $\|g\| \rightarrow \infty$. Indeed, let $\tilde{x}, \tilde{x}' \in D$ such that $\tilde{\pi}\tilde{x} = x, \tilde{\pi}\tilde{x}' = x'$, let $\check{\zeta}: [0, 1] \rightarrow \tilde{M}$ be a straight line such that $\check{\zeta}(0) = \tilde{x}'$, $\check{\zeta}(1) = \tilde{x}$ and denoted by ζ the projection of $\check{\zeta}$ down to M , we obviously have that

$$h_g^{k+1}(x, x) \leq h_g^k(x, x') + [A(\zeta)]$$

holds for each g . We thus verify the claim as $[A(\zeta)]$ is a finite number.

Proposition 2.5. *There exists positive number $a > 0$ such that $h_g^\infty(x, x') \geq \|g\|a$ holds for each $(x, x') \in \mathbb{T}^n \times \mathbb{T}^n$ and for sufficiently large $\|g\|$.*

Proof. If the conclusion is not true, for each small number $\epsilon_i > 0$, there would be suitably large $k_i, T_i \in \mathbb{N}$ such that

$$\frac{1}{k_i \|g\|} h_g^{T_i}(x, x') < \epsilon_i.$$

By assumption, $k_i, T_i \rightarrow \infty$ as $\epsilon_i \rightarrow 0$. Let γ_i be the minimizer realizing the quantity $h_g^{T_i}(x, x')$, let $\mu_i = (\gamma, \dot{\gamma})^* \nu_i$ where ν_i is a probability measure evenly distributed on the interval $[0, T_i]$. By the weak*-compactness, some probability measure μ and subsequence $i_j \rightarrow \infty$ exist such that $\mu_{i_j} \rightharpoonup \mu$. Clearly, μ is invariant for the Lagrange flow, $\int L d\mu = 0$ and $\rho(\mu) = \lambda g \in H_1(M, U, \mathbb{R})$. But it is absurd. \square

2.3. Globally elementary weak KAM solutions. For the configuration manifold \mathbb{T}^n , each weak KAM solution is 1-periodic in x_i for $i = 1, 2, \dots, n$, where $(x_1, x_2, \dots, x_n) = x$ denotes the configuration coordinate. If a finite covering of \mathbb{T}^n is considered to be configuration space, weak KAM solution may not be 1-periodic for each coordinate.

We assume that the minimal measure contains finitely many ergodic components $\mu_c^1, \mu_c^2, \dots, \mu_c^m$ for each cohomology class, it is a generic phenomenon [BC]. In this case, elementary weak KAM solution for each μ_c^i is well-defined. The lift of μ_c^i to a finite covering $k\mathbb{T}^n$ may contain several connected components. For instance, if $\mathcal{M} \subset \{|x_1| \leq \delta\} \times \mathbb{T}^{n-1}$, then there are two connected components in the lift of \mathcal{M} for $2\mathbb{T} \times \mathbb{T}^{n-1}$. However, there are cases that the minimal measure is always uniquely ergodic for any finite covering manifold, an example is KAM torus.

Given $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ with $k_i \geq 1$ for each $i = 1, 2, \dots, n$, we define an equivalence relation \sim_k in \mathbb{R}^n : we say $x \sim_k x'$ if $x_i - x'_i = 2jk_i$ for some $j \in \mathbb{Z}$ ($i = 1, 2, \dots, n$). Clearly, $\pi_k: M_k = \mathbb{R}^n / \sim_k \rightarrow \mathbb{T}^n$ is a finite covering of \mathbb{T}^n . In the following, we shall also use the symbols: $\pi_{\infty, k}: \mathbb{R}^n \rightarrow M_k$ and $\pi_{\infty}: \mathbb{R}^n \rightarrow \mathbb{T}^n$ to denote the projection. For a bounded domain $\Omega \subset \mathbb{R}^n$, if the topology of $\pi_{\infty, k}\Omega \subset M_k$ is trivial, we use the same symbol to denote its projection $\Omega := \pi_{\infty, k}\Omega$.

Let \mathcal{M}_{∞} and \mathcal{M}_k be the lift of Mather set \mathcal{M} to the universal covering space as well as to M_k respectively. The connected components are denoted by $\mathcal{M}_{\infty, i}$ and $\mathcal{M}_{k, i}$ correspondingly. Obviously, the unit cube $D = [0, 1]^n$ intersects finitely many connected components of \mathcal{M}_{∞} , denoted by $\mathcal{M}_{\infty, i}$ with $i = 0, 1, \dots, m$.

Let $d_k = \min\{k_1, k_2, \dots, k_n\}$. Some $R_D > 0$ exists such that for any $k \in \mathbb{Z}^n$ with $d_k \geq R_D$, $\pi_{\infty, k}\mathcal{M}_{\infty, i} \neq \pi_{\infty, k}\mathcal{M}_{\infty, j}$ holds for $0 \leq i, j \leq m$ and $i \neq j$. In this case, we number $\mathcal{M}_{k, i}$ such that $\mathcal{M}_{k, j} = \pi_{\infty, k}\mathcal{M}_{\infty, j}$ for $0 \leq j \leq m$. Let $u_{k, j}^{\pm}$ denote the elementary weak KAM for $\mathcal{M}_{k, j}$ with respect to the configuration manifold M_k .

Lemma 2.1. *For each bounded region $\Omega \subset \mathbb{R}^n$, there exists $R_{\Omega} > 0$ such that for any $k, k' \in \mathbb{Z}^n$ with $d_k, d_{k'} \geq \max\{R_{\Omega}, R_D\}$,*

$$u_{k, j}^{\pm}|_{\Omega} = u_{k', j}^{\pm}|_{\Omega} + \text{constant}$$

holds for each $j = 0, 1, \dots, m$.

Proof. We only need to consider the case that the minimal measure is uniquely ergodic. If there are finitely many ergodic components, we obtain this result by perturbing the Lagrangian so that it is uniquely ergodic and applying Theorem 2.1.

Each weak KAM solution for \mathbb{T}^n is a weak KAM solution for any M_k . If the lift of the minimal measure to any finite covering space is still uniquely ergodic, the elementary weak KAM solution remains the same.

Let us consider the case that there are more than one connected component in \mathcal{M}_k with $d_k \geq R_D$. Remember $\mathcal{M}_{k, j} = \pi_{\infty, k}\mathcal{M}_{\infty, j}$ for $0 \leq j \leq m$ where $\mathcal{M}_{\infty, j}$ intersects the fundamental domain $[0, 1]^n$. Considered as a function defined in \mathbb{R}^n , the elementary weak KAM solution $u_{k, j}^{-}$ determined by $\mathcal{M}_{k, j}$ is k_i -periodic in the i -th coordinate. Because of the definition of elementary weak KAM solution, a sequence of functions $u_{k, j, \epsilon}^{-}$ exists such that $u_{k, j, \epsilon}^{-} \rightarrow u_{k, j}^{-}$ as $\epsilon \rightarrow 0$, where $u_{k, j, \epsilon}^{-}$ is the weak

KAM solution for the Lagrangian $L_{k,\epsilon} : TM_k \rightarrow \mathbb{R}$. This Lagrangian satisfies the following conditions:

- 1, it is the same as L when it is restricted on the tangent bundle of a neighborhood U of $\mathcal{M}_{k,j}$, i.e. $L_{k,\epsilon}|_{TU} = L|_{TU}$;
- 2, the minimal measure is uniquely ergodic whenever $\epsilon \neq 0$, supported on $\mathcal{M}_{k,j}$;
- 3, $L_{k,\epsilon} \rightarrow L$ as $\epsilon \rightarrow 0$.

Starting from each $x \in M_k$, there exists at least one backward semi-static curve for $L_{k,\epsilon}$, $\gamma_{k,x,\epsilon} : (-\infty, 0]$ with $\gamma_{k,x,\epsilon}(0) = x$. Clearly, $\pi\alpha(d\gamma_{k,x,\epsilon}) \cap \mathcal{M}_{k,j} \neq \emptyset$. Let $t_i \rightarrow \infty$ be the sequence so that $\gamma_{k,x,\epsilon}(-t_i) \rightarrow x_0 \in \mathcal{M}_{k,j}$, let α stand for the average action, then we have

$$(2.5) \quad u_{k,\epsilon}^-(x) - u_{k,\epsilon}^-(x_0) = \lim_{t_i \rightarrow \infty} \int_{-t_i}^0 L_{k,\epsilon}(d\gamma_{k,x,\epsilon}(s))ds + t_i\alpha.$$

Let $D_k = \{\bar{x} : \bar{x}_i \in [-k_i, k_i]\} \subset \mathbb{R}^n$ so that $\pi_{\infty,k} : D_k \rightarrow M_k$ is an injection and $\pi_{\infty,k}D_k = M_k$. Let $\bar{x} \in D_k$ be the points such that $\pi_{\infty,k}\bar{x} = x$. Let $\bar{\gamma}_{k,\bar{x},\epsilon}$ be the lift of $\gamma_{k,x,\epsilon}$ to the universal covering space so that $\bar{\gamma}_{k,\bar{x},\epsilon}(0) = \bar{x}$. It is possible that $\pi\alpha(d\bar{\gamma}_{k,\bar{x},\epsilon}) \cap \mathcal{M}_{\infty,j} \neq \emptyset$. The curve may approach to another connected component of \mathcal{M}_{∞} . Let $\Omega_d = \{x : \max_i |x_i| \leq d\} \subset \mathbb{R}^n$. Note that $L_{k,\epsilon}$ is a small perturbation of L . In virtue of Proposition 2.5 we claim that $\bar{\gamma}_{k,\bar{x},\epsilon}$ approaches to $\mathcal{M}_{\infty,0}$ provided $\bar{x} \in \Omega_d$, ϵ is suitably small and d_k is sufficiently large. Let us assume the contrary, i.e. $\bar{\gamma}_{k,\bar{x},\epsilon}$ approaches to another connected component of \mathcal{M}_{∞} . In this case, $\|[\pi_k \gamma_{k,\bar{x},\epsilon}]\|$ would be sufficiently large provided d_k is sufficiently large. By Proposition 2.5 the action of L along $\pi_k \gamma_{k,\bar{x},\epsilon}$

$$\int L(d\pi_k \gamma_{k,\bar{x},\epsilon}(t), t)dt \geq \|\pi_k \gamma_{k,\bar{x},\epsilon}\|a$$

with $a > 0$. As $L_{k,\epsilon}$ is a small perturbation of L , the action of $L_{k,\epsilon}$ along $\pi_k \gamma_{k,\bar{x},\epsilon}$ would approach infinity as $d_k \rightarrow \infty$. The absurdity verifies the claim.

The set $\{\dot{\gamma}_{k,x,\epsilon}(0)\}$ is compact as $\epsilon \rightarrow 0$. For each accumulation point v , there is a subsequence of $\epsilon \rightarrow 0$ such that $\dot{\gamma}_{k,x,\epsilon}(0) \rightarrow v$. The initial value (x, v) uniquely determines an orbit $(\gamma_{k,x}, \dot{\gamma}_{k,x})$ of L . The curve $\gamma_{k,x} : (-\infty, 0] \rightarrow M_k$ is a backward semi-static curve for L which may not approach to $\mathcal{M}_{k,0}$. When $\epsilon \rightarrow 0$, $\gamma_{k,x,\epsilon}$ may approach not only one but a family of semi-static curves for L including the curves connecting different connected components of \mathcal{M}_k . More precisely, there might be several connected components $\mathcal{M}_{k,i_0} = \mathcal{M}_{k,0}, \mathcal{M}_{k,i_1}, \dots, \mathcal{M}_{k,i_l}$ and semi-static curves $\gamma_{j,j+1}$ of L for M_k ($j = 0, 1, \dots, l-1$) such that $\pi\alpha(d\gamma_{j,j+1}) \cap \mathcal{M}_{k,i_j} \neq \emptyset$, $\pi\omega(d\gamma_{j,j+1}) \cap \mathcal{M}_{k,i_{j+1}} \neq \emptyset$, $\pi\alpha(d\gamma_{k,x}) \cap \mathcal{M}_{k,i_l} \neq \emptyset$ and each $\gamma_{j,j+1}$ is approached by $\gamma_{k,x,\epsilon}$ as $\epsilon \rightarrow 0$. These curves have their natural projection down to \mathbb{T}^n , denoted by the same symbol.

We define the quantity $A_{i,j} : \mathcal{M}_i \times \mathcal{M}_j \rightarrow \mathbb{R}$

$$(2.6) \quad [A_{i,j}(x_i, x_j)] = \inf_{\substack{\gamma(0)=x_i \\ \gamma(k)=x_j \\ k \in \mathbb{Z}_+}} \int_0^k L(d\gamma(s))ds + k\alpha.$$

By definition of weak KAM, for almost every point x , $(x, \partial_x u_{k,\epsilon}^-(x))$ uniquely determines a backward semi-static curve $\gamma_{k,x,\epsilon}$. Since this semi-static curve approached to

several curves: $\gamma_{k,x,\epsilon} \rightarrow \gamma_{1,2} * \cdots * \gamma_{i-1,i} * \gamma_{k,x}$ we obtain that

$$(2.7) \quad \begin{aligned} u_{k,0}^-(x) - u_{k,0}^-(x_0) &= \lim_{t_i \rightarrow \infty} \int_{-t_i}^0 L(d\gamma_x(s)) ds + t_i \alpha \\ &\quad + \sum_{i=0}^{i-1} [A_{j_i, j_{i+1}}(x_i, x_{i+1})] \end{aligned}$$

where $t_i \rightarrow \infty$ is a sequence such that $\gamma_{k,x}(-t_i) \rightarrow x_i \in \mathcal{M}_{k,i_i}$, $x_j \in \mathcal{M}_{k,i_j}$.

From the argument above, we can see the following. For each $x \in \Omega_d$, the backward semi-static curve $\gamma_{k,x,\epsilon}$ approaches to $\mathcal{A}_{k,0}$ provided d_k is sufficiently large. For different k, k' satisfying this condition, $\gamma_{k,x,\epsilon}$ and $\gamma_{k',x,\epsilon}$ may converge to different curves, $\gamma_{k,x,\epsilon} \rightarrow \gamma_{1,2} * \cdots * \gamma_{i-1,i} * \gamma_{k,x}$ and $\gamma_{k',x,\epsilon} \rightarrow \gamma'_{1,2} * \cdots * \gamma'_{i'-1,i'} * \gamma'_{k,x}$ as $\epsilon \rightarrow 0$. But the action of L along $\gamma_{1,2} * \cdots * \gamma_{i-1,i} * \gamma_{k,x}$ is the same as along $\gamma'_{1,2} * \cdots * \gamma'_{i'-1,i'} * \gamma'_{k,x}$. Indeed, let \tilde{k} be an integer vector divisible both to k and k' , i.e. $\tilde{k} = mk = m'k'$ for $m, m' \in \mathbb{Z}$. We consider the lift of $\gamma_{k,x,\epsilon}$ as well as $\gamma_{k',x,\epsilon}$, denoted by $\tilde{\gamma}_{k,x,\epsilon}$ and $\tilde{\gamma}_{k',x,\epsilon}$ with $\tilde{\gamma}_{k,x,\epsilon}(0) = \tilde{\gamma}_{k',x,\epsilon}(0)$. The action of $L_{k,\epsilon}$ along $\tilde{\gamma}_{k,x,\epsilon}$ is almost the same as the action of $L_{k',\epsilon}$ along $\tilde{\gamma}_{k',x,\epsilon}$ as the perturbation is sufficiently small. Thus, we obtain from the formula 2.7 that

$$\bar{u}_{k,0}|_{\Omega} = \bar{u}_{k',0}|_{\Omega} + \text{constant}$$

if both d_k and $d_{k'}$ are sufficiently large. \square

Definition 2.2. *The function $\bar{u}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is called globally elementary weak KAM solution for $\mathcal{M}_{\infty,i}$ if for each bounded domain $\Omega \subset \mathbb{R}^n$, there exists $R_{\Omega} > 0$ such that for any $k \in \mathbb{Z}^n$ with $d_k \geq R_{\Omega}$,*

$$\bar{u}_{k,i}|_{\Omega} = \bar{u}_i|_{\Omega} + \text{constant}$$

holds for each elementary weak KAM solution $\bar{u}_{k,i} : M_k \rightarrow \mathbb{R}$ for $\mathcal{M}_{k,i}$.

From Lemma 2.1, we obtain the existence of a globally elementary weak KAM solution for each $\mathcal{M}_{\infty,i}$.

To investigate the properties of globally elementary weak KAM solution, let us consider a special case first, i.e. the Aubry class contains a set homeomorphic to \mathbb{T}^{n-1} . In this case, each $\mathcal{A}_{\infty,i}$ divided \mathbb{R}^n into two parts, denoted by R^- and R^+ .

Theorem 2.5. *If the Aubry class contains a set homeomorphic to \mathbb{T}^{n-1} , then the globally elementary weak KAM solution $\bar{u}_i^{\pm} : \mathbb{R}^n \rightarrow \mathbb{R}$ has a decomposition*

$$\bar{u}_i^{\pm} = v_i^{\pm} + w_i^{\pm},$$

where v_i^{\pm} is periodic and w_i^{\pm} is affine when they are restricted in the half space R^+ as well as in another half space R^- .

Proof. We only need to consider the case that the minimal measure is uniquely ergodic, as we did in the proof of Lemma 2.1. According to Theorem 2.4, there are exactly two types of minimal homoclinic orbits to $\tilde{\mathcal{A}}$, we pick up two representative elements $\gamma_-, \gamma_+ : \mathbb{R} \rightarrow \mathbb{T}^n$. Let

$$h_{\pm} = \liminf_{t_i^{\pm} \rightarrow \infty} \int_{-t_i^{\pm}}^{t_i^{\pm}} L(d\gamma_{\pm}(t), t) dt + (t_i^- + t_i^+) \alpha,$$

where t_i^{\pm} is chosen such that $\gamma_{\pm}(t_i^+) \rightarrow 0$ and $\gamma_{\pm}(-t_i^-) \rightarrow 0$.

As the Aubry class is co-dimension one, we are able to number all connected components by \mathcal{A}_i ($i = \dots -1, 0, 1, 2, \dots$) such that any path from \mathcal{A}_{i-1} to \mathcal{A}_{i+1} must pass through \mathcal{A}_i . Denote by Π_i the strip bounded by \mathcal{A}_i and \mathcal{A}_{i+1} . \mathcal{A}_0 separates \mathbb{R}^n into two parts, denoted by D^- and D^+ such that $\mathcal{A}_{-1} \subset D^-$ and $\mathcal{A}_1 \subset D^+$.

Let $\bar{\gamma}_\pm$ denote the lift of γ_\pm to \mathbb{R}^n such that $\alpha(d\bar{\gamma}_\pm) \subset \tilde{\mathcal{A}}_{\infty,0}$. Thus, we have either $\omega(d\bar{\gamma}_{k,-}) \subset \tilde{\mathcal{A}}_{k,-1}$ and $\omega(d\bar{\gamma}_{k,+}) \subset \tilde{\mathcal{A}}_{k,1}$, or $\omega(d\bar{\gamma}_{k,+}) \subset \tilde{\mathcal{A}}_{k,-1}$ and $\omega(d\bar{\gamma}_{k,-}) \subset \tilde{\mathcal{A}}_{k,1}$. We only need to study one case, let's say, the first case.

Given a bounded domain $\Omega \subset \mathbb{R}^n$. From the definition of globally elementary weak KAM solution, we see that

$$\bar{u}_0^-|_\Omega = u_{k,0}^-|_\Omega$$

whenever d_k is suitably large. Clearly, it is periodic when it is restricted $\Omega \cap \Pi_i$, i.e. $u_{k,0}^-(x) = u_{k,0}^-(x')$ if $x' - x \in \mathbb{Z}^n$ and $x, x' \in \Omega \cap \Pi_i$.

For each $x \in \Omega \cap \Pi_i$ with $i > 0$, there exists at least one point $x_0 \in \Omega \cap \Pi_0$ such that $x - x_0 \in \mathbb{Z}^n$. By definition, we find that $u_{k,0}^-(x) = u_{k,0}^-(x_0) + ih_+$. Obviously, $u_{k,0}^-(x) = u_{k,0}^-(x_0) + (1+i)h_-$ if $x \in \Omega \cap \Pi_i$ with $i < 0$.

Pick up a point $x_0 \in \mathcal{A}_{\infty,0}$. For each non-zero integer vector $k \in \mathbb{Z}^n$, the point $x = x_0 + k$ stays in certain $\mathcal{A}_{\infty,i}$. Along the ray $x = x_0 + tk$ with $t > 0$, we define

$$v_0^-(x) = \begin{cases} u_0^-(x) - u_0^-(x_0) - tih_+, & \text{if } i > 0; \\ u_0^-(x) - u_0^-(x_0) - tih_-, & \text{if } i < 0, \end{cases}$$

Clearly, $u_0^- - v_0^-$ is affine and v_0^- is periodic when they are restricted in D^- as well as in D^+ . \square

Given an ergodic component of a minimal measure with higher co-dimensions, it is unclear what condition guarantees the decomposition of the globally elementary weak KAM solutions. It appears closely related to the problem whether there are infinitely many types of minimal homoclinic orbits to the Aubry class.

Proposition 2.6. *Let u_i^\pm be the globally elementary weak KAM solution for \mathcal{M}_i . Then, for $g \in H_1(\mathcal{A}_i, \mathbb{Z})$ u_i^\pm remains bounded on the whole ray $\{x_0 + tg : t \in \mathbb{R}_+\}$, where $\mathcal{A}_i \supset \mathcal{M}_i$ is an Aubry class; for $g \in H_1(M, \mathcal{A}, \mathbb{Z})/K$, u_i^\pm grows up linearly, or asymptotically linearly on the ray $\{x_0 + tg : t \in \mathbb{R}_+\}$.*

Proof. For arbitrarily large t , there exists $x \in \mathcal{M}_{\infty,i}$ such that $\text{dist}(x_0 + tg, x) \leq 2$ and $\mathcal{M}_{\infty,i} \cap D \neq \emptyset$ where D is the unit cube containing the origin. Let $x^* = \pi_\infty x$, $\bar{\xi}$ be a curve connecting x^* to x , $\xi = \pi_\infty \bar{\xi}$, then $[\xi] \in H_1(\mathcal{A}, \mathbb{Z})$. By definition,

$$\inf_{[\xi] \in H_1(\mathcal{A}, \mathbb{Z})} \inf_{\substack{\xi(0) = \xi(k) \\ k \in \mathbb{Z}_+}} \int_0^k L(d\xi(t), t) dt = 0$$

it proves the first conclusion.

For the second, one can see from Proposition 2.4 that it grows up at least linearly. Given $g \in H_1(M, \mathcal{A}_i, \mathbb{Z})/K$ finitely many elements $g_0, g_1, \dots, g_r \in H_1(\mathbb{T}^n, \mathbb{Z})$ exists

such that for each $g = \sum_{i=0}^r j_i g_i$ with $j_i \in \mathbb{Z}_+$. Thus,

$$h_g^\infty(x, x) \leq \sum_{i=0}^r j_i h_{g_i}^\infty(x, x).$$

For each $\pi_\infty x \in \mathcal{M}_i$, $[x - x^*] = g$, we have

$$u(x) - u(x^*) = h_g^\infty(\pi_\infty x, \pi_\infty x).$$

This completes the proof. \square

3. DYNAMICS AROUND FIXED POINT

Given a Tonelli Lagrangian $L: T\mathbb{T}^n \rightarrow \mathbb{R}$, let $c_0 = \arg \min \alpha$. Any minimal measure with zero-rotation vector must be c_0 -minimal measure. In this section we study the dynamics around the Mather set for the class c_0 . The motivation comes from following argument.

Let us consider the normal form of a nearly integrable Hamiltonian

$$H(p, q) = h(p) + \epsilon P(p, q), \quad (p, q) \in \mathbb{R}^d \times \mathbb{T}^d.$$

around a complete resonant point. Let $\omega(y) = \nabla h(y)$ denote the frequency vector of the unperturbed system. A frequency ω is called complete resonant of (minimal) period T if $T\omega \in \mathbb{Z}^d$ and $t\omega \notin \mathbb{Z}^d$ for each $t \in (0, T)$. By finitely many steps of KAM iteration and one step of linear coordinate transformation on torus, one obtains a normal form of nearly integrable Hamiltonian (see Appendix A)

$$\tilde{H}(\tilde{x}, \tilde{y}) = \tilde{h}(\tilde{y}) + \epsilon \tilde{Z}(x, \tilde{y}) + \epsilon \tilde{R}(\tilde{x}, \tilde{y})$$

where $\tilde{x} = (x, x_d)$, $\tilde{y} = (y, y_d)$, $(x, y) \in \mathbb{T}^{d-1} \times \mathbb{R}^{d-1}$, \tilde{H} is well-defined in $(\tilde{x}, \tilde{y}) \in \mathbb{T}^d \times B_d(\tilde{y}^*)$, $\partial \tilde{h}(\tilde{y}^*) = (0, \omega_d)$ and $\epsilon \tilde{R}$ is a higher order term.

Since $\partial_{y_d} \tilde{h}(\tilde{y}^*) = \omega_d \neq 0$, there exists some function $G(x, y, \tau)$ solving the equation $\tilde{H}(x, -\tau, y, G(x, y, \tau)) = E$ provided $E > \min \alpha_{\tilde{H}}$, which defines a time-periodic Hamiltonian system with n -degrees of freedom. Here $\tau = -x_d$ plays the role of time. Clearly, one can write

$$G(x, y) = h(y) + \epsilon Z(x, y) + \epsilon R(x, y, \tau)$$

where ϵR is a higher order term of ϵ and $\partial h(y^*) = 0$, i.e. the complete resonance reduces to zero frequency. Omitting the higher order term, one obtains Hamiltonian with d degrees of freedom

$$\bar{G}(x, y) = h(y) + \epsilon Z(x, y).$$

It determines a Lagrangian we shall study in this section.

3.1. Flat of the α -function. By definition, a subset is called a flat of certain α -function if, restricted on this set, the α -function is affine, and no longer affine on any set properly containing this set. As α -function is convex with super-linear growth, each flat is a convex and bounded set. Given an n -dimensional flat \mathbb{F} , a subset in $\partial \mathbb{F}$ is called an edge if it is contained in a $(n - 1)$ -dimensional hyperplane. Since flat is convex, edge is also convex.

Theorem 3.1. *Given a class $c_0 \in H^1(\mathbb{T}^n, \mathbb{R})$, if the minimal measure is uniquely ergodic, supported on a hyperbolic fixed point, then there exists an n -dimensional flat $\mathbb{F}_0 \subset H^1(\mathbb{T}^n, \mathbb{R})$ such that this point supports a c -minimal measure for all $c \in \mathbb{F}_0$.*

Remark: The condition of this theorem does not exclude topological non-triviality of the Aubry set. An example is the product of n pendulums. The Aubry set covers the whole torus \mathbb{T}^n if the Lagrangian L is replaced by $L - \langle c, \dot{x} \rangle$ with c being on the boundary of the flat.

Proof. By translation one can assume that the fixed point is at $(x, \dot{x}) = (0, 0)$, by adding a closed 1-form and a constant to the Lagrangian, one can assume $c_0 = 0$ and $L(0, 0) = 0$. In this case, for any finite $T > 0$ we have

$$\inf_{\gamma(-T)=\gamma(T)} A(\gamma) = \inf_{\gamma(-T)=\gamma(T)} \int_{-T}^T L(d\gamma(t)) dt > 0,$$

otherwise, there would be another minimal measure.

To each closed curve $\xi: [-T, T] \rightarrow \mathbb{T}^n$ with $\xi(-T) = \xi(T)$ a first homology class $[\xi] = h \in H(\mathbb{T}^n, \mathbb{Z})$ is associated. We consider the quantity

$$A(h) = \liminf_{T \rightarrow \infty} \inf_{\substack{\xi(-T)=\xi(T) \\ [\xi]=h}} \int_{-T}^T L(d\xi(t)) dt \geq 0.$$

There exist at least $n+1$ irreducible classes $h_i \in H(\mathbb{T}^n, \mathbb{Z})$ and $n+1$ minimal homoclinic orbits $d\gamma_i$ such that $A([h_i]) = A(\gamma_i)$ [Be1]. Clearly, $H_1(\mathbb{T}^n, \mathbb{Z})$ can be generated by the homology classes of all minimal homoclinic curves over \mathbb{Z}_+ .

We abuse the notation h to denote homology class $h \in H_1(\mathbb{T}^n, \mathbb{Z})$ or to denote a point $h \in \mathbb{Z}^n$. For each curve $\bar{\gamma}_T: [-T, T] \rightarrow \mathbb{R}^n$ with $\bar{\gamma}_T(-T) = 0$ and $\bar{\gamma}_T(T) = h$, Clearly,

$$A(h) = \liminf_{T \rightarrow \infty} \inf_{\substack{\bar{\gamma}_T(-T)=0 \\ \bar{\gamma}_T(T)=h}} \int_{-T}^T L(d\bar{\gamma}_T(t)) dt.$$

Recall the definition of elementary weak-KAM and note that the point $x = 0$ is the support of the minimal measure. Let u_0^- (u_0^+) denote the backward (forward) globally elementary weak-KAM based on the set $\{x = 0\}$, we have

$$A(h) = u_0^-(h) - u_0^-(0), \quad A(-h) = u_0^+(0) - u_0^+(h).$$

By setting $u_0^-(0) = u_0^+(0)$, we claim

$$(3.1) \quad A(h) + A(-h) = u_0^-(h) - u_0^+(h) > 0.$$

The quantity $A(h)$ is achieved may not by a curve connecting the origin to $h \in \mathbb{Z}^n$, but may by the conjunction of several curves $\bar{\gamma}_1 * \bar{\gamma}_2 * \cdots * \bar{\gamma}_m$. Let $\bar{\gamma}_i: \mathbb{R} \rightarrow \mathbb{R}^n$ denote a curve ($i = 1, \dots, m$), the conjunction implies that $\bar{\gamma}_i(-\infty) = \bar{\gamma}_{i-1}(\infty)$. Let $\gamma_i = \pi_n \bar{\gamma}_i$, where $\pi_n: \mathbb{R}^n \rightarrow \mathbb{T}^n$ denotes the standard projection. In this case, $\gamma_1, \dots, \gamma_m$ are minimal homoclinic curves such that $h = \sum_{i=1}^m [\gamma_i]$. By the definition of elementary weak-KAM, each $\bar{\gamma}_i$ is a (u_0^-, L) -calibrated curve. Let $h_i = \sum_{j=1}^i [\gamma_j]$. Obviously, some large $t_0 > 0$ exists such that $(\bar{\gamma}_i(t), \dot{\bar{\gamma}}_i(t))$ stays in the local stable manifold of $(h_i, 0)$ whenever $t \geq t_0$. Therefore, some constant C_i exists such that

$$(3.2) \quad u_0^-(\bar{\gamma}_i(t)) = u_{h_i}^+(\bar{\gamma}_i(t)) + C_i, \quad \forall t \geq t_0,$$

where we use $u_{h_i}^-$ and $u_{h_i}^+$ to denote the globally elementary-KAM based on $x = h_i$. Clearly, $u_{h_i}^+$ and $u_{h_i}^-$ generate local stable and unstable manifold around the point $(x, \dot{x}) = (h_i, 0)$ respectively.

Because that $u_{h_i}^\pm$ is L -dominate function, for suitably small $\delta > 0$, the following holds

$$u_{h_i}^+(x) - u_{h_i}^+(h_i) \leq u_0^\pm(x) - u_0^\pm(h_i) \leq u_{h_i}^-(x) - u_{h_i}^-(h_i),$$

for $x \in B_\delta(h_i)$ (see Theorem 5.1.2 in [Fa2]). Remember that $u_0^- \geq u_0^+$. If

$$A(h) + A(-h) = u_0^-(h) - u_0^+(h) = 0,$$

one obtains from (3.2) that some t_0 exists such that

$$u_0^+(\bar{\gamma}_m(t)) = u_0^-(\bar{\gamma}_m(t)), \quad \forall t \geq t_0.$$

Since u_0^+ is an L -dominate function and $\bar{\gamma}_i$ is a (u_0^-, L) -calibrated curve for each $1 \leq i \leq m$,

$$\begin{aligned} u_0^+(\bar{\gamma}_i(t_0)) - u_0^+(\bar{\gamma}_i(t_1)) &\leq \int_{t_1}^{t_0} L(d\bar{\gamma}_i(s))ds, \\ u_0^-(\bar{\gamma}_i(t_0)) - u_0^-(\bar{\gamma}_i(t_1)) &= \int_{t_1}^{t_0} L(d\bar{\gamma}_i(s))ds \end{aligned}$$

hold for any $t_1 \leq t_0$. This induces $u_0^-(\bar{\gamma}_m(t)) = u_0^+(\bar{\gamma}_m(t))$ for all $t \in \mathbb{R}$, and induces in turn the inequality for $i = m-1, m-2, \dots$, and finally we have

$$u_0^-(\bar{\gamma}_1(t)) = u_0^+(\bar{\gamma}_1(t)), \quad \forall t \in \mathbb{R}.$$

Since $u_0^-(x) > u_0^+(x)$ holds in a small neighborhood of 0

$$u_0^-(\bar{\gamma}_1(t)) = u_0^+(\bar{\gamma}_1(t)), \quad \forall t \in \mathbb{R}.$$

On the other hand, as the fixed point $\{x = 0\}$ is hyperbolic, some $\delta > 0$ exists such that

$$u_0^-(x) - u_0^+(x) > 0, \quad \forall x \in B_\delta(0) \setminus \{0\},$$

if we set $u_0^-(0) = u_0^+(0)$. This contradiction proves the formula (3.1).

Let

$$\mathbb{G}_0 = \{h \in H_1(\mathbb{T}^n, \mathbb{Z}) : \exists \gamma : \mathbb{R} \rightarrow \mathbb{T}^n \text{ s.t. } [\gamma] = h, A(\gamma) = 0\}.$$

\mathbb{G}_0 is said to generate a rational direction $h \in \mathbb{Z}^n$ over \mathbb{Z}_+ if there exist $k, k_i \in \mathbb{Z}_+$ and $h_i \in \mathbb{G}_0$ such that

$$kh = \sum k_i h_i.$$

It is an immediate consequence of the formula (3.1) that once \mathbb{G}_0 generates a rational direction $h \in \mathbb{Z}^n$ over \mathbb{Z}_+ , then it can not generate the direction $-h$ over \mathbb{Z}_+ . Therefore, the set

$$\text{span}_{\mathbb{R}_+} \mathbb{G}_0 = \{\sum a_i h_i : h_i \in \mathbb{G}_0, a_i \geq 0\}$$

is a cone properly restricted in half space. Thus, there exists an n -dimensional cone \mathbb{C}_0 such that

$$\langle c, h \rangle > 0, \quad \forall c \in \mathbb{C}_0, h \in \text{span}_{\mathbb{R}_+} \mathbb{G}_0.$$

Since the minimal measure for zero cohomology class is supported on the fixed point, $\tilde{\mathcal{N}}(0)$ is composed by those minimal homoclinic orbits along which the action equals zero. According to the upper semi-continuity of Mañé set in cohomology class, any minimal measure μ_c is supported by a set lying in a small neighborhood

of these homoclinic orbits provided $|c|$ is very small. Consequently, we have $\rho(\mu_c) \in \text{span}_{\mathbb{R}_+} \mathbb{G}_0$, where $\rho(\mu_c)$ denotes the rotation vector of μ_c .

Let us consider a cohomology class c such that $-c \in \mathbb{C}_0$ and $|c| \ll 1$. We claim that the c -minimal measure is also supported on the fixed point. Indeed, if it is not true, we would have positive average action of L : $A(\mu_c) > 0$, since the minimal measure for zero class is assumed unique and supported on the fixed point. From the choice of c we see that $\langle c, \rho(\mu_c) \rangle < 0$. Thus, one obtains

$$A_c(\mu_c) = A(\mu_c) - \langle c, \rho(\mu_c) \rangle > 0 = A_c(\mu),$$

it deduces absurdity. For this c , the action of the Lagrangian $L_c = L - \langle c, \dot{x} \rangle$ along any minimal homoclinic curve γ is positive,

$$A(\gamma) - \langle c, [\gamma] \rangle > 0,$$

namely, the Aubry set for this class is also a singleton. Consequently, $\mu_{c'}$ is also supported on this point provided c' sufficiently close to c . This verifies the existence of an n -dimensional flat. \square

Eigenvalues of the fixed point

Let us consider the eigenvalues of the fixed point by assuming the hyperbolicity, denoted by λ_i ($i = 1, 2, \dots, 2k$). Under the hyperbolic assumption, half of these have positive real part, other half have negative real part. In general, these eigenvalues may have non-zero imaginary part. But in nearly integrable systems, all eigenvalues are real.

Proposition 3.1. *If the Lagrangian is a small perturbation of integrable one $L = \ell(\dot{x}) + \epsilon P(x, \dot{x})$ where ℓ is positive definite in \dot{x} , then for generic P and for sufficiently small ϵ , all eigenvalues at the fixed point are real and different.*

Proof. Let $A = \partial_{\dot{x}\dot{x}}^2 L$, $B = \partial_{x\dot{x}}^2 P$, $C = \partial_{xx}^2 P$ evaluated at the fixed point. As the minimal measure is supported on a fixed point, C is positive definite. We consider the linearized equation and assume the solution with the form of $x = \xi \exp \sqrt{\epsilon} \lambda t$, then

$$(3.3) \quad |\lambda^2 A - \sqrt{\epsilon} \lambda (B - B^t) - C|_{k \times k} = 0.$$

Let $A_0 = \partial_{\dot{x}\dot{x}}^2 \ell$, evaluated at the fixed point. For generic C , all solutions of the equation

$$(3.4) \quad |\lambda^2 A_0 - C|_{k \times k} = 0.$$

are real and different from each other: $\lambda = \pm \lambda_1, \pm \lambda_2, \dots, \pm \lambda_k$, $\lambda_i \neq \lambda_j$ if $i \neq j$. Since (3.3) is a small perturbation of (3.4), all solutions of (3.3) are different, and consequently, real. If there was a complex solution $\lambda = \sigma + i\omega$, $\pm \sigma \pm i\omega$ would be solution also, which is guaranteed by the Hamiltonian structure. It implies the existence of more than $2k$ solutions, the absurdity induces the proposition. \square

The shape of the flat

Here, we are concerned about the flat $\mathbb{F}_0 = \mathcal{L}_\beta(0)$. It is a n -dimensional flat if the c -minimal measure is supported on the hyperbolic fixed point for each $c \in \text{int} \mathbb{F}_0$. By coordinate translation, we assume it is at the origin: $(\dot{x}, x) = (0, 0) = (x, y)$. Let (ξ_i^\pm, η_i^\pm) denote the eigenvector for $\pm \lambda_i$, where ξ_i^\pm is for the x -coordinates, η_i^\pm is for the y -coordinates. We assume

- 1, all eigenvalues are real number and different;
- 2, all minimal homoclinic curves approach to the fixed point in the direction ξ_1^\pm as $t \rightarrow \mp\infty$.

The condition 1 is obviously generic. To see the genericity of the condition 2, let us remind reader that there is, generically, at most one minimal homoclinic curve for each homology class. By further perturbation, it approaches to the fixed point in the direction of ξ_1^\pm . Since there are countably many homology classes, the genericity is obtained.

For $\theta > 0$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we define a cone

$$C(\xi, \theta) = \{x \in \mathbb{R}^n : |\langle x, \xi \rangle| \geq \theta \|\xi\| \|x\|\},$$

and let

$$C(\xi, \theta, d) = \{x \in C(\xi, \theta) : \|x\| = d\}.$$

Proposition 3.2. *Assume that $(x, y) = (0, 0) \in \{H^{-1}(0)\}$ is a hyperbolic fixed point for Φ_H^t , where all eigenvalues are real and different:*

$$\text{Spec}\{J\nabla H\} = \{\pm\lambda_1, \dots, \pm\lambda_n; \quad 0 < \lambda_1 < \dots < \lambda_n\}.$$

Let (ξ_i^\pm, η_i^\pm) denote the eigenvector for $\pm\lambda_i$, where ξ_i^\pm is for the x -coordinates, η_i^\pm is for the y -coordinates. Let $(x(t), y(t)) \subset \{H^{-1}(0)\}$ be an orbit such that $x(t)$ passes through a ball $B_\delta(0) \subset \mathbb{R}^n$, $x(-T) \in \partial B_\delta(0)$, $x(T) \in \partial B_\delta(0)$ and $x(t) \in \text{int} B_\delta(0)$ for all $t \in (-T, T)$. Then, for suitably small $\delta > 0$ and $\theta = \frac{1}{2}$, there exist sufficiently large $T_0 > 0$ such that for $T \geq T_0$ one has

$$(x(-T), x(T)) \notin C(\xi_1^-, \theta, \delta) \times C(\xi_1^+, \theta, \delta).$$

Proof. Without losing of generality we assume that the fixed point is at the origin. By certain symplectic coordinate transformation, the Hamiltonian is assumed to have the normal form

$$H(x, y) = \sum_{i=1}^n \frac{1}{2} (y_i^2 - \lambda_i^2 x_i^2) + P(x, y)$$

where $P = O(\|(x, y)\|^3)$ is a higher order term. By the method of variation of constants, we obtain the solution of the corresponding Hamilton equation

$$(3.5) \quad \begin{aligned} x_i(t) &= e^{-\lambda_i t} (b_i^- + F_i^-) + e^{\lambda_i t} (b_i^+ + F_i^+), \\ y_i(t) &= -\lambda_i e^{-\lambda_i t} (b_i^- + F_i^-) + \lambda_i e^{\lambda_i t} (b_i^+ + F_i^+), \end{aligned}$$

where b_i^\pm are constants determined by boundary condition and

$$\begin{aligned} F_i^- &= \frac{1}{2\lambda_i} \int_0^t e^{\lambda_i s} (\lambda_i \partial_{y_i} P + \partial_{x_i} P)(x(s), y(s)) ds, \\ F_i^+ &= \frac{1}{2\lambda_i} \int_0^t e^{-\lambda_i s} (\lambda_i \partial_{y_i} P - \partial_{x_i} P)(x(s), y(s)) ds. \end{aligned}$$

Substituting (x, y) with the formula (3.5) in the Hamiltonian we obtain a constraint for the constants b_i^\pm :

$$(3.6) \quad H(x(t), y(t)) = -2 \sum_{i=1}^n \lambda_i^2 b_i^- b_i^+ + P((b_i^+ + b_i^-), \lambda_i(b_i^+ - b_i^-))$$

Let us estimate the size of the constants c_i^\pm by the boundary conditions $x(T) = (x_1^+, x_2^+, \dots, x_k^+) \in \partial B_\delta(0)$, $x(-T) = (x_1^-, x_2^-, \dots, x_k^-) \in \partial B_\delta(0)$ and assuming

$$(3.7) \quad \min\{|x_1^-|, |x_1^+|\} \geq \frac{\delta}{2}.$$

For suitably small $\theta > 0$, $(x(-T), x(T)) \in C(\xi_1^-, \theta, \delta) \times C(\xi_1^+, \theta, \delta)$ implies (3.7) holds. Since the fixed point $(x, y) = 0$ is hyperbolic, the curve given by (3.5) stays entirely in the ball $B_\delta(0)$: $x(t) \in \text{int} B_\delta(0)$ provided $t \in (-T, T)$ with suitably large $T > 0$. Since the curve $x|_{[-T, T]}$ stays inside of the ball $B_\delta(0)$ and T is sufficiently large, the orbit $(x, y)|_{[-T, T]}$ stays near the stable and unstable manifold of the fixed point, i.e. it stays in $B_\delta(0)$. Note $P = O(\|(x, y)\|^3)$, we obtain from the theorem of Grobman-Hartman that

$$\begin{aligned} x_i^- &= b_i^- e^{\lambda_i T} + b_i^+ e^{-\lambda_i T} + o(\delta), \\ x_i^+ &= b_i^- e^{-\lambda_i T} + b_i^+ e^{\lambda_i T} + o(\delta). \end{aligned}$$

for sufficiently large $T > 0$. In this case, it deduces from the assumption (3.7) that

$$|b_1^\pm| \geq \frac{\delta}{3} e^{-\lambda_1 T},$$

and

$$|b_i^\pm| \leq 2\delta e^{-\lambda_i T}, \quad \forall i = 2, \dots, k.$$

Since $\lambda_1 < \lambda_i$ for each $i \geq 2$, $|b_i^\pm| \ll |b_1^\pm|$ if T is sufficiently large. In this case, we obtain from (3.6) that

$$|H(x(t), y(t))| > |\lambda_1^2 b_1^+ b_1^-| > 0.$$

It contradicts the assumption that $(x(t), y(t)) \in \{H^{-1}(0)\}$, thus completes the proof. \square

This proposition tells us a fact: in the energy level $\{H^{-1}(0)\}$ there does not exist such an orbit passing through $B_\delta(0)$ in the direction close to ξ_1^\pm .

Theorem 3.2. *Let $\mathbb{F}_0 = \mathcal{L}_\beta(0)$ be an n -dimensional flat of the α -function. Each minimal homoclinic curve γ is assumed approaching to the fixed point in the direction of the eigenvectors corresponding to the smallest eigenvalue*

$$\lim_{t \rightarrow \pm\infty} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{\xi_1^\mp}{\|\xi_1^\mp\|}.$$

and all eigenvalues are assumed real and different. It is also assumed that, for each $c \in \mathbb{F}_0$ (including the boundary), the minimal measure is uniquely supported on a hyperbolic fixed point, then, there are only finitely many homological types of minimal homoclinic orbits, namely, a finite set

$$H_{\mathbb{F}_0} = \{h_1, h_2, \dots, h_m\}$$

exists such that $[\gamma] \in H_{\mathbb{F}_0}$ provided γ is a minimal homoclinic curve. Consequently, the flat \mathbb{F}_0 is a polygon with finitely many edges, denoted by $\mathbb{E}_1, \dots, \mathbb{E}_m$ such that $\mathcal{A}(c)$ is composed by minimal homoclinic curves with homological type h_i if c is in the interior of \mathbb{E}_i and

$$\langle c - c', h_i \rangle = 0, \quad \forall c, c' \in \mathbb{E}_i.$$

Proof. As the minimal measure is uniquely ergodic and supported on a point for each class in \mathbb{F}_0 , what could be contained in the Mañé set are homoclinic orbits and the point itself.

We claim that the Mañé set contains at least one homoclinic orbit for each $c \in \partial\mathbb{F}_0$. Otherwise, for some class $c' \notin \mathbb{F}_0$ with $\|c - c'\|$ being small, the homology of the Mañé set is trivial, the same as that for c . It is guaranteed by the upper semi-continuity of Mañé set in cohomology class. In this case, we have $\langle c, \rho(\mu_c) \rangle = \langle c', \rho(\mu'_c) \rangle = 0$, consequently,

$$-\alpha(c') = A(\mu_{c'}) - \langle c', \rho(\mu'_c) \rangle \geq A(\mu_c) = -\alpha(c).$$

On the other hand, we have $\alpha(c') > \alpha(c)$ as $c' \notin \mathbb{F}_0$. The contradiction verifies our claim.

Conversely, it is obvious that each minimal homoclinic curve γ stays in certain Aubry set:

$$\cup_{t \in \mathbb{R}} \gamma(t) \subset \mathcal{A}(c), \quad \forall c \in \lim_{\delta \searrow 0} \mathcal{L}_\beta(\delta[\gamma]) \subset \mathbb{F}_0.$$

where the limit is in the sense of Hausdorff and \mathcal{L}_β represents the Fenchel-Legendre transformation $H_1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$.

If $H_{\mathbb{F}_0}$ contains infinitely many elements, there would be infinitely many minimal homoclinic curves $\gamma_1, \gamma_2 \cdots \gamma_k \cdots$ such that $[\gamma_i] \neq [\gamma_j]$ provided $i \neq j$. Thus we have two possibilities.

1, a neighborhood $B_d(0)$ of the fixed point exists such that each minimal homoclinic curve γ hits $\partial B_d(0)$ exactly twice, i.e. $\exists t^- < t^+$ such that $\gamma(t) \in B_d(0)$ for all $t \in (-\infty, t^-] \cup [t^+, \infty)$ and $\gamma(t) \notin B_d(0)$ for all $t \in (t^-, t^+)$;

2, for any small $d > 0$, there are infinitely many minimal homoclinic curves $\gamma_{i_1}, \gamma_{i_2} \cdots$ with different homology classes which pass through the sphere $\partial B_d(0)$ in finite time, i.e. $\exists t^- < t_0^- < t_0^+ < t^+$ such that $\gamma(t) \in B_d(0)$ for all $t \in (-\infty, t^-] \cup [t_0^-, t_0^+] \cup [t^+, \infty)$ and $\gamma(t) \notin B_d(0)$ holds for some $t \in (t^-, t_0^-)$ as well as for some $t \in (t_0^+, t^+)$.

Let us study the first possibility. Denote by $t_i^- < t_i^+$ the time when the minimal homoclinic curve γ_i hits the sphere $\partial B_d(0)$. For each γ_i , there is a segment $\gamma_i|_{(t_i^-, t_i^+)}$ staying outside of $B_d(0)$. Each $d\gamma_i|_{(t_i^-, t_i^+)}$ generates a probability measure μ_i on $T\mathbb{T}^n$ such that

$$\int f d\mu_i = \frac{1}{|t_i^+ - t_i^-|} \int_{t_i^-}^{t_i^+} f(d\gamma_i(s)) ds$$

holds for each continuous function f . As all these curves have different homology class, $\|[\gamma_i]\| \rightarrow \infty$ as $i \rightarrow \infty$. Note the speed along all these curves are uniformly bounded, we have

$$|t_i^+ - t_i^-| \rightarrow \infty, \quad \text{as } i \rightarrow \infty.$$

Denote by $c_i \in \partial\mathbb{F}_0$ such that γ_i stays in the Aubry class for c_i and let $c^* \in \partial\mathbb{F}_0$ be an accumulation point of $\{c_i\}$, then some c^* -minimal measure μ^* exists such that $\mu_i \rightarrow \mu^*$. Obviously, μ^* is not supported on the fixed point, and it contradicts the assumption that the minimal measure is always uniquely ergodic for each $c \in \mathbb{F}_0$.

Let us study the second possibility. In this case, for suitably small $\delta > 0$, there would be an infinite sequence of homoclinic curves γ_i and correspondingly the sequence of time $t_i^- < t_i^+$ such that $\gamma_i(t) \in B_\delta(0)$ for each $t \in [t_i^-, t_i^+]$, $\gamma_i(t_i^\pm \mp \epsilon) \notin B_\delta(0)$ and $|t_i^+ - t_i^-| \rightarrow \infty$ as $i \rightarrow \infty$. By using Proposition 3.2, we find that some $\theta > 0$ exists such that one of the inequalities in the following holds for each i

$$(3.8) \quad \left\| \frac{\dot{\gamma}_i(t_i^-)}{\|\dot{\gamma}_i(t_i^-)\|} - \frac{\xi_1^+}{\|\xi_1^+\|} \right\| > \theta, \quad \left\| \frac{\dot{\gamma}_i(t_i^+)}{\|\dot{\gamma}_i(t_i^+)\|} - \frac{\xi_1^-}{\|\xi_1^-\|} \right\| > \theta$$

provided i is sufficiently large. It implies that there exists some minimal homoclinic curve γ as well as some eigenvector $\xi_{k_1}^-$ or $\xi_{k_2}^+$ with $k_1 \neq 1$ and $k_2 \neq 1$ such that at least one of the following holds

$$\lim_{t \rightarrow -\infty} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{\xi_{k_2}^+}{\|\xi_{k_2}^+\|}, \quad \lim_{t \rightarrow \infty} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{\xi_{k_1}^-}{\|\xi_{k_1}^-\|}.$$

But this contradicts to the condition. Therefore, $H_{\mathbb{F}_0}$ contains finitely many elements only.

Let γ, γ' be two minimal homoclinic curves contained in the Aubry set $\mathcal{A}(c)$, $\mathcal{A}(c')$ respectively. Let $\Gamma = \{\xi c + (1 - \xi)c' : \xi \in [0, 1]\}$. If Γ intersects the interior of \mathbb{F}_0 , then $[\gamma] \neq [\gamma']$. Indeed, by definition we have

$$\begin{aligned} A(\gamma) - \langle c, [\gamma] \rangle &= 0, & A(\gamma') - \langle c, [\gamma'] \rangle &\geq 0; \\ A(\gamma') - \langle c', [\gamma'] \rangle &= 0, & A(\gamma) - \langle c', [\gamma] \rangle &\geq 0, \end{aligned}$$

it follows that $A(\gamma) = A(\gamma')$ provided $[\gamma] = [\gamma']$. Consequently,

$$\begin{aligned} 0 &= \xi(A(\gamma) - \langle c, [\gamma] \rangle) + (1 - \xi)(A(\gamma') - \langle c', [\gamma'] \rangle) \\ &= A(\gamma) - \langle \xi c + (1 - \xi)c', [\gamma] \rangle \\ &= A(\gamma') - \langle \xi c + (1 - \xi)c', [\gamma'] \rangle. \end{aligned}$$

On the other hand, the Aubry set for each class in the interior of \mathbb{F}_0 contains the fixed point only. The contradiction implies that $[\gamma] \neq [\gamma']$.

This deduces that \mathbb{F}_0 is a polygon with exactly m edges, each edge corresponds to exactly one homology type of minimal homoclinic curve. \square

Recall $\partial\mathbb{F}_0$ is a topological sphere with $n - 1$ -dimension. If there is an $(n - 1)$ -dimensional open disk $U^{n-1} \subset \mathbb{F}_0$ where all conditions in the last theorem hold, then from the proof of the theorem, one can see that, for any small $\delta > 0$, the conclusion of that theorem holds in $U^{n-1} - \delta$.

3.2. Modulus of continuity in terms of energy. Let γ_0 be a minimal homoclinic curve approaching to the fixed point in the direction of ξ^\pm corresponding to the smallest eigenvalue $\pm\lambda_1$, let \mathbb{E}_0 be a sub-flat of \mathbb{F}_0 . In this subsection we assume that

- 1, for each $c \in \mathbb{F}_0$, the Mather set contains exactly one fixed point $(x, \dot{x}) = (0, 0)$;
- 2, for each $c \in \text{int}\mathbb{E}_0$, the Aubry set consists of the fixed point and the minimal homoclinic curve $\mathcal{A}(c) = \cup_{t \in \mathbb{R}} \gamma_0(t) \cup \{0\}$;

3, there exists a sequence of positive numbers $\nu_i > 0$ with $\nu_i \rightarrow 0$ as $i \rightarrow \infty$ such that some ergodic minimal measure μ_i exists such that $\rho(\mu_i) = \nu_i[\gamma_0]$, and $\mathcal{A}(c) = \mathcal{M}(c)$ for each $c \in \mathcal{L}_\beta(\nu_i[\gamma_0])$.

By definition, there exists an elementary weak KAM for each μ_i . It is uniquely related to the energy $E_i = \alpha(\mathcal{L}_\beta(\nu_i[\gamma_0]))$. The main purpose of this section is to study the modulus of continuity of some functions in terms of energy at $E = 0$.

Dependence of the average speed on energy.

According to Birkhoff's ergodic theorem, there is an orbit $d\zeta_i: \mathbb{R} \rightarrow \mathbb{T}^n$ of ϕ_L^t such that

$$\frac{1}{2T}A(\zeta_i|_{[-T,T]}) \rightarrow A(\mu_i), \quad \frac{1}{2T}(\bar{\zeta}_i(T) - \bar{\zeta}_i(-T)) \rightarrow \rho(\mu_i)$$

as $T \rightarrow \infty$. Where $\bar{\zeta}_i$ stands for a lift of ζ_i to the universal covering space. By the upper semi-continuity of Mañé set, the curve ζ_i passes through the ball $B_\delta(0)$ infinitely many times provided ν_i is suitably small. Denoted by $t_{i,k}^+$ and $t_{i,k}^-$ the time when ζ_i enters and departs the ball respectively, i.e $\zeta_i(t) \in B_\delta(0)$ for each $t \in [t_{i,k}^+, t_{i,k}^-]$, $\zeta_i(t_{i,k}^\pm \mp \delta) \notin B_\delta(0)$. Clearly,

$$|t_{i,k}^- - t_{i,k}^+| \rightarrow \infty, \quad \text{as } E_i \rightarrow 0.$$

If ζ_i is a periodic curve, there is some $t_i > 0$ such that $t_{i,k+1}^+ = t_{i,k}^- + t_i$ holds for each $k \in \mathbb{Z}$.

Let $t^+, t^- \in \mathbb{R}$ such that the minimal homoclinic curve γ_0 enters $B_\delta(0)$ at $t = t^+$ and goes out of $B_\delta(0)$ at $t = t^-$. By the upper semi-continuity of Mañé set one has

$$(\zeta_i(t_{i,k}^\pm), \dot{\zeta}_i(t_{i,k}^\pm)) \rightarrow (\gamma_0(t^\pm), \dot{\gamma}_0(t^\pm)).$$

Since each minimal homoclinic curve approaches to the fixed point in the direction corresponding to smallest eigenvalue,

$$(3.9) \quad \left\| \frac{\dot{\zeta}_i(t_{i,k}^\pm)}{\|\dot{\zeta}_i(t_{i,k}^\pm)\|} - \frac{\xi_1^\mp}{\|\xi_1^\mp\|} \right\| < \frac{1}{4}$$

holds provided $\delta > 0$ is suitably small and $t_{i,k}^- - t_{i,k}^+$ is suitably large.

Each segment $\zeta_i|_{[t_{i,k}^+, t_{i,k}^-]}$ is of course a solution of the Hamilton equation. In the coordinates of normal form, it is given by Eq. (3.5) and the integral constants c_i^\pm satisfy the constraint (3.6). The condition (3.9) induces

$$\frac{1}{2}\delta e^{-\lambda_1|t_{i,k}^- - t_{i,k}^+|/2} \leq |b_1^\pm| \leq 2\delta e^{-\lambda_1|t_{i,k}^- - t_{i,k}^+|/2}.$$

For small $\delta > 0$ and sufficiently large $|t_{i,k}^- - t_{i,k}^+|$, we have

$$|b_j^\pm| \leq 2\delta e^{-\lambda_j|t_{i,k}^- - t_{i,k}^+|/2}, \quad \forall j = 2, \dots, n,$$

$$|P((b_j^+ + b_j^-), \lambda_j(b_j^+ - b_j^-))| \leq a_1 e^{-3\lambda_1|t_{i,k}^- - t_{i,k}^+|/2}$$

where the constant a_1 depends only on the function P itself, the constant b_j is introduced in Eq.(3.5) as integral constants to determine the solution of certain ODE.

Thus, we obtain from (3.6) that

$$\begin{aligned}
E_i &= \left| -2 \sum_{j=1}^n \lambda_j^2 b_i^+ b_i^- + P((b_j^+ + b_j^-), \lambda_j(b_j^+ - b_j^-)) \right| \\
&\geq \frac{1}{2} \lambda_1^2 \delta^2 e^{-\lambda_1 |t_{i,k}^- - t_{i,k}^+|} - 8 \sum_{j=2}^n \lambda_j^2 \delta^2 e^{-\lambda_j |t_{i,k}^- - t_{i,k}^+|} - a_1 e^{-3\lambda_1 |t_{i,k}^- - t_{i,k}^+|/2} \\
&\geq \frac{1}{4} \lambda_1^2 \delta^2 e^{-\lambda_1 |t_{i,k}^- - t_{i,k}^+|}
\end{aligned}$$

provided $\delta > 0$ is suitably small and $|t_{i,k}^- - t_{i,k}^+|$ is sufficiently large. Under the same condition, E_i is obviously upper bounded by

$$\begin{aligned}
E_i &= \left| -2 \sum_{j=1}^n \lambda_j^2 b_i^+ b_i^- + P((b_j^+ + b_j^-), \lambda_j(b_j^+ - b_j^-)) \right| \\
&\leq 8 \lambda_1^2 \delta^2 e^{-\lambda_1 |t_{i,k}^- - t_{i,k}^+|} + 8 \sum_{j=2}^n \lambda_j^2 \delta^2 e^{-\lambda_j |t_{i,k}^- - t_{i,k}^+|} + a_1 e^{-3\lambda_1 |t_{i,k}^- - t_{i,k}^+|/2} \\
&\leq 9 \lambda_1^2 \delta^2 e^{-\lambda_1 |t_{i,k}^- - t_{i,k}^+|}.
\end{aligned}$$

Therefore, we find the dependence of speed on the energy

$$(3.10) \quad |t_{i,k}^- - t_{i,k}^+| = \frac{1}{\lambda_1} |\ln E_i| - \frac{2}{\lambda_1} |\ln \delta| + \tau_{i,k}$$

where $\tau_{i,k}$ is uniformly bounded for each $k \in \mathbb{Z}$:

$$\frac{1}{\lambda_1} (2 \ln \lambda_1 - 2 \ln 2) \leq \tau_{i,k} \leq \frac{1}{\lambda_1} (2 \ln \lambda_1 + 3 \ln 3).$$

Obviously, $t_{i,k+1}^+ - t_{i,k}^- \rightarrow t^+ - t^-$ as $i \rightarrow \infty$, some $E_0 > 0$ exists such that

$$\frac{1}{2} (t^+ - t^-) \leq t_{i,k+1}^+ - t_{i,k}^- \leq 2(t^+ - t^-), \quad \text{if } E_i \leq E_0.$$

Consequently, one has

$$(3.11) \quad \left| \frac{1}{\nu_i} - \frac{1}{\lambda_1} |\ln \delta^{-2} E_i| \right| \leq \tau_0$$

where $\tau_0 = \frac{1}{\lambda_1} |\ln 9 \lambda_1^2| + 2(t^+ - t^-)$.

3.3. Around the two-dimensional flat. In this section we restrict ourselves to the special case that $k = 2$. The task of this section is to study the structure of the Mather sets as well as of the Mañé sets in a neighborhood of the resonant point. Under the coordinate transformation (A.10), it corresponds to a fixed point.

Recall that the Aubry set for one cohomology class is a Lipschitz graph over the configuration manifold [Ma2]. Because that the manifold is two dimensional, each orbit in the Aubry has to be *parallel* to any other orbit in the support in the sense that these curves do not intersect each other. On the other hand, for autonomous system, $\beta(\lambda\omega)$, regarded as the function of $\lambda \in \mathbb{R}$, is differentiable at each $\lambda \neq 0$ (see [Ms]). Thus, we have

Proposition 3.3. *Assume that L is an autonomous Tonelli Lagrangian defined on \mathbb{T}^2 . For each non-zero rational vector, the Mather set consists of periodic orbits with the same rotation vector.*

Each minimal measure with zero-rotation corresponds to the minimum of the α -function. There are two *nondegenerate* cases for the set of minimal point, denoted by $\mathbb{F}_0 \subset H^1(\mathbb{T}^2, \mathbb{R})$. We call a case *nondegenerate* if it persists under small perturbation.

1, \mathbb{F}_0 is a two-dimensional flat. Typically, for each class in the interior of \mathbb{F}_0 , the minimal measure is supported on a fixed point, or a shrinkable periodic orbit $(\gamma, \dot{\gamma})$, i.e. $[\gamma] = 0$. This fixed point (periodic orbit) is of hyperbolic type.

2, \mathbb{F}_0 is one-dimensional. Typically, the minimal measure is supported on two periodic orbits $(\gamma_-, \dot{\gamma}_-)$ and $(\gamma_+, \dot{\gamma}_+)$ with the property:

$$[\gamma_-]/\|[\gamma_-]\| = -[\gamma_+]/\|[\gamma_+]\|.$$

Both orbits are non-degenerate, i.e. hyperbolic.

\mathbb{F}_0 can not be a singleton, since the β function can not have a two-dimensional flat when the configuration space is two-dimensional torus and the Lagrangian is autonomous,

Let us study the first case. We assume that some cohomology class c_0 is in the interior of \mathbb{F}_0 , the point $(x, \dot{x}) = (0, 0)$ supports the minimal measure. Clearly, $\mathcal{A}(c) = \{0\}$ holds for each $c \in \text{int}\mathbb{F}_0$. The study is similar if it is supported on a shrinkable closed orbit.

Given the α as well as β -function, let us recall the Fenchel-Legendre transformation $\mathcal{L}_\beta: H_1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ is defined as

$$\mathcal{L}_\beta(\omega) = \{c : \alpha(c) + \beta(\omega) = \langle c, \omega \rangle\}.$$

Let

$$\partial^*\mathbb{F}_0 = \{c \in \partial\mathbb{F}_0 : \mathcal{M}(c) \setminus \{x_0\} \neq \emptyset\}, \quad \Omega_{\mathbb{F}_0} = \mathcal{L}_\beta^{-1}(\partial^*\mathbb{F}_0),$$

it may be non-empty. Let us give an example:

$$L = \frac{1}{2}\dot{x}_1^2 + \frac{\lambda^2}{2}\dot{x}_2^2 + V(x)$$

where $|\lambda| \neq 1$, the potential satisfies the following conditions: $x = 0$ is the only minimal point of V ; there exist two numbers $d > d' > 0$ such that

$$\int_0^1 V(\gamma(s))ds \geq d$$

holds for any closed curve $\gamma: [0, 1] \rightarrow \mathbb{T}^2$ passing through the origin and $[\gamma] \neq 0$; restricted a neighborhood of circle $x_2 = a$ with $a \neq 0 \bmod 1$, $V = d' + (x_2 - a)^2$. By calculation, we find that $\partial\mathbb{F}_0 \cap \{c_2 = 0\} = \{c_1 = \pm\sqrt{2d'}\}$. Indeed,

$$L \pm c_1\dot{x}_1 = \frac{1}{2}(x_1 \pm c_1)^2 + \frac{\lambda^2}{2}\dot{x}_2^2 + V(x) - \frac{1}{2}c_1^2,$$

thus, the Mather set for $c = (\pm\sqrt{2d'}, 0)$ is composed by a fixed point $x = 0$ and a periodic curve $x(t) = (x_{1,0} \pm \sqrt{2d'}t, a)$.

Clearly, $\partial^*\mathbb{F}_0$ is closed with respect to \mathbb{F}_0 . Each $\omega \in \Omega_{\mathbb{F}_0} \cap H_1(\mathbb{T}^2, \mathbb{Q})$ determines unique $\lambda_\omega > 0$ and unique indivisible element $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ such that $\omega = \lambda_\omega g$. Let

$$h_g^T = \inf_{\substack{\xi(0)=\xi(T) \\ [\xi]=g}} \int_0^T L(d\xi(s))ds,$$

and γ_g^T denote the minimizer of h_g^T . For each $g = \lambda_\omega^{-1}\omega$ with $\omega \in \Omega_{\mathbb{F}_0}$, there exists $T_g > 0$ such that $\cup_{t \in [0, T]}(\gamma_g^T(t), \dot{\gamma}_g^T(t))$ supports a minimal measure for certain $c \in H^1(\mathbb{T}^2, \mathbb{R}) \setminus \mathbb{F}_0$ provided $T < T_g$ and $\gamma_g^{T_g}$ determines an ergodic minimal measure with rotation vector ω for certain $c \in \partial^* \mathbb{F}_0$.

If there exists a minimal homoclinic orbit $(\gamma, \dot{\gamma})$ such that $[\gamma] = g$, the action of L along this curve would be

$$[A(\gamma)|_{\mathbb{R}}] = \liminf_{T \rightarrow \infty} h_g^T.$$

Clearly, if there does not exist $\omega \in \Omega_{\mathbb{F}_0}$ such that $g = \lambda_\omega^{-1}\omega$, then

$$\liminf_{T \rightarrow \infty} h_g^T = \inf_T h_g^T.$$

For the co-homology class $c \in \partial^* \mathbb{F}_0$, the existence of infinitely many \bar{M} -minimal homoclinic orbits has been proved in [Zhe, Zo1]. Along these orbits one get closer and closer to the closed orbit. Obviously, these orbits are associated different homologies. However, these orbits are not in Aubry set for any $c \in \mathbb{F}_0$. Indeed, the action along each of these orbits are positive: $\liminf_{T \rightarrow \infty} h_g^T - \langle c, g \rangle > 0$. When $\Omega_{\mathbb{F}_0} = \emptyset$, there are at least three minimal homoclinic orbits to the fixed point. With a minimal homoclinic orbit $d\gamma: \mathbb{R} \rightarrow M$, one can associate an element $[\gamma] \in H_1(\mathbb{T}^2, \mathbb{Z})$.

The existence of homoclinic orbit to some Aubry set is closely related to the existence of certain flat.

Lemma 3.1. *Given $c, c' \in \mathbb{F}$, let $c_\lambda = \lambda c + (1 - \lambda)c'$. Then*

$$\tilde{\mathcal{A}}(c) \cap \tilde{\mathcal{A}}(c') = \tilde{\mathcal{A}}(c_\lambda), \quad \forall \lambda \in (0, 1).$$

Proof. Using argument in [Ms], for any curve $\gamma: \mathbb{R} \rightarrow M$, we have

$$[A_{c^*}(\gamma|_I)] = \lambda[A_c(\gamma|_I)] + (1 - \lambda)[A_{c'}(\gamma|_I)], \quad \forall I \subset \mathbb{R}.$$

The result is induced. \square

Lemma 3.2. *For each cohomology class, the Aubry set is assumed containing finitely many classes. Let \mathbb{F}_0 be a 2-dimensional flat, the Mather set is a singleton for each class in the interior of \mathbb{F}_0 , let \mathbb{E}_i be a sub-flat of \mathbb{F}_0 , then*

$$\mathcal{A}(c') \supsetneq \mathcal{A}(c)$$

holds for $c' \in \partial \mathbb{F}_0 \setminus (\partial \mathbb{E}_i)$ and $c \in \text{int} \mathbb{F} \setminus (\text{int} \mathbb{E}_i)$ respectively.

Proof. Aubry set is upper semi-continuous in cohomology class if it contains finitely many classes for each cohomology [Be3]. For each $c' \in \partial \mathbb{F}_0$, some $c^* \notin \mathbb{F}_0$ exists arbitrarily close to c' . $\mathcal{M}(c^*)$ contains some points far away from x_0 . Let $c^* \rightarrow c'$, we see $\mathcal{A}(c') \supsetneq \{x_0\}$.

Let us consider sub-flat \mathbb{E}_i . If $\mathcal{M}(c)$ contains only a point for $c \in \mathbb{E}_i$, the Aubry set would contains a homoclinic orbit $d\zeta$. In this case, $\exists k_i^\pm \rightarrow \infty$ such that $\|\zeta(-k_i^-) - \zeta(k_i^+)\| \rightarrow 0$ and $[A_c(\zeta|_{[-k_i^-, k_i^+]})] \rightarrow 0$. Some other $\hat{c} \in \mathbb{E}_i$ if and only $[A_{\hat{c}}(\zeta|_{[-k_i^-, k_i^+]})] \rightarrow 0$. It follows that $\langle c - \hat{c}, [\zeta] \rangle = 0$. If $\mathcal{M}(c)$ contains an invariant curve ζ , in the same way we also obtain $\langle c - \hat{c}, [\zeta] \rangle = 0$ if $\hat{c} \in \mathbb{E}_i$.

Let $c' \in \partial \mathbb{E}_i$ and $c \in \text{int} \mathbb{E}_i$. Some other $c^* \in \partial \mathbb{F}_0 \setminus \mathbb{E}_i$ exists arbitrarily close to c' . As the line connecting c to c^* passes through the interior of \mathbb{F}_0 , we obtain from Lemma

3.1 that $\mathcal{A}(c) \cap \mathcal{A}(c^*)$ consists of only one point. Clearly, we have $\langle c - c^*, [\zeta] \rangle \neq 0$, otherwise, ζ would be contained in $\mathcal{A}(c^*)$ too. As c^* is also on the boundary of \mathbb{F}_0 , there exists an invariant curve or a homoclinic curve $\xi \subset \mathcal{A}(c^*)$. We claim $[\zeta] \neq [\xi]$. Indeed, if $d\zeta$ is a homoclinic orbit and ξ is an invraint orbit with period T , we have

$$\int_{-\infty}^{\infty} L(d\zeta)dt - \langle c, [\zeta] \rangle = 0, \quad \int_0^T L(d\xi)dt - \langle c^*, [\xi] \rangle = 0.$$

If $[\zeta] = [\xi]$, we would obtain

$$\int_{-\infty}^{\infty} L(d\zeta)dt \neq \int_0^T L(d\xi)dt.$$

Consequently, as $\alpha(c) = \alpha(c^*)$, we would obtain either

$$\int_{-\infty}^{\infty} L(d\zeta)dt - \langle c^*, [\zeta] \rangle < 0,$$

or

$$\int_0^T L(d\xi)dt - \langle c, [\xi] \rangle < 0.$$

But it is absurd. Therefore, some $x^* \in \mathcal{A}(c^*)$ remains far away from $\mathcal{A}(c)$. Let $c^* \rightarrow c'$, one obtains $\mathcal{A}(c') \supsetneq \mathcal{A}(c)$. The proof is the same if ξ is a homoclinic curve. \square

Recall the definition of G_m in the section 2: a first homology class $g \in G_m$ if and only if \exists minimal homoclinic orbit $d\gamma$ such that $[\gamma] = g$. Let $G_{m,c} \subset G_m$ be defined such that $g \in G_{m,c}$ if and only if \exists minimal homoclinic orbit $d\gamma$ in $\tilde{\mathcal{A}}(c)$ such that $[\gamma] = g$. We say that there are k -types of minimal homoclinic orbits in $\tilde{\mathcal{A}}(c)$ if $G_{m,c}$ contains exactly k elements. For a sub-flat E_i we define $G_{m,\mathbb{E}_i} = G_{m,c}$ for some $c \in \text{int}\mathbb{E}_i$, from the proof of Lemma 3.2 one can see that it makes sense.

Theorem 3.3. *We assume that, for each cohomology class, the Aubry set contains finitely many classes. Let \mathbb{F}_0 be a two dimensional flat, $\mathcal{M}(c_0)$ is a singleton for $c_0 \in \text{int}\mathbb{F}_0$. Let \mathbb{E}_i denote certain sub-flat of \mathbb{F}_0 , then*

- 1, *either $\mathbb{E}_i \cap \partial^*\mathbb{F}_0 = \emptyset$ or $\mathbb{E}_i \subset \partial^*\mathbb{F}_0$;*
- 2, *if $\mathbb{E}_i \cap \partial^*\mathbb{F}_0 = \emptyset$, then G_{m,\mathbb{E}_i} contains exactly one element;*
- 3, *if $c \in \partial\mathbb{E}_i$ and $c \notin \partial^*\mathbb{F}_0$ then $G_{m,c}$ contains exactly two elements;*
- 4, *if $\mathbb{E}_i, \mathbb{E}_j \subset \partial^*\mathbb{F}_0$, then either \mathbb{E}_i and \mathbb{E}_j are disjoint, or $\mathbb{E}_i = \mathbb{E}_j$;*
- 5, *if $\mathbb{E}_i \subset \partial^*\mathbb{F}_0$, $\mathcal{M}(c) = \mathcal{M}(c')$ holds for $c \in \partial\mathbb{E}_i$ and $c' \in \text{int}\mathbb{E}_i$.*

Proof. For conclusion 1, let us recall a fact that $\mathcal{A}(c) = \mathcal{A}(c')$ provided $c, c' \in \text{int}\mathbb{E}_i$ [Ms]. If $\exists c \in \text{int}\mathbb{E}_i$ such that the Aubry set consists of minimal homoclinic orbit and the fixed point only, it is then the case for all classes in the interior of \mathbb{E}_i . Clearly, for any $c, c' \in \mathbb{E}_i$ and each $g \in G_{m,\mathbb{E}_i}$ we have $\langle g, c - c' \rangle = 0$. Thus, we only need to consider $c \in \partial\mathbb{E}_i$. Let us assume the contrary, namely, some ergodic invariant measure μ_c exists which is not supported on the singleton and minimizes the action

$$\int Ld\mu_c - \langle \rho(\nu_c), c \rangle = -\alpha(c).$$

Because the configuration space is two-dimensional torus, the rotation vector of the measure $\rho(\mu_c)$ must be parallel to $g \in G_{m,\mathbb{E}_i}$, otherwise the Lipschitz graph property of

Aubry set will be violated. Consequently, $\langle \rho(\mu_c), c - c' \rangle = 0$ holds for each $c' \in \text{int}\mathbb{E}_i$. It implies that this measure also minimizes the action for $c' \in \text{int}\mathbb{E}_i$, but it is absurd.

As the configuration space is two dimensional torus, its genus equals one, $G_{m,c}$ contains at most two elements. It is due to the Lipschitz graph property of Aubry set. If G_{m,\mathbb{E}_i} contains two elements, due to Lemma 3.2, $G_{m,c}$ would contain more than at least three elements if $c \in \partial\mathbb{E}_i$ and $c \in \partial^*\mathbb{F}_0$, or there would be an extra closed orbit in $\mathcal{A}(c)$. In both cases, the Lipschitz graph property of Aubry set would be violated. Conclusion 2, 3 are the immediate consequence.

If the conclusion 4 is not true, then for the cohomology class in $\mathbb{E}_i \cap \mathbb{E}_j$ the Mather set would contain two closed circles with different homology, but it violate the Lipschitz graph property of Aubry set. Due to the same reason we obtain the conclusion 5. \square

By this theorem, for each sub-flat $\mathbb{E}_i \subset \partial^*\mathbb{F}_0$ a class $\rho(\mathbb{E}_i)$ is uniquely determined which is the rotation vector of the minimal measure for $c \in \text{int}\mathbb{E}_i$. For brevity, we also use the notation $\mathcal{M}(\mathbb{E}_i) = \mathcal{M}(c)$ for $c \in \mathbb{E}_i$.

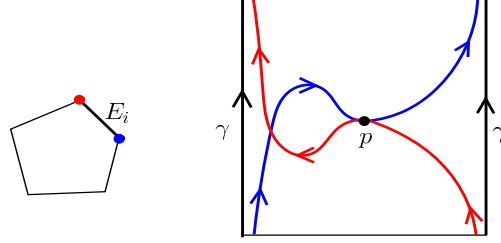


FIGURE 1. $\mathbb{E}_i \subset \partial^*\mathbb{F}_0$, $\mathcal{M}(\mathbb{E}_i) = \{0\} \cup \{\gamma\}$. The blue curve is in $\mathcal{A}(c)$ for c at one end point of \mathbb{E}_i , the red curve is in $\mathcal{A}(c')$ for c' at another end point of \mathbb{E}_i .

Given two homology classes $g, g' \in H_1(\mathbb{T}^2, \mathbb{Z})$, we call them adjacent if $g \in G_{m,\mathbb{E}}$, $g' \in G_{m,\mathbb{E}'}$, $\mathbb{E} \cap \partial^*\mathbb{F}_0 = \emptyset$, $\mathbb{E}' \cap \partial^*\mathbb{F}_0 = \emptyset$, \mathbb{E} and \mathbb{E}' are adjacent. The special topology of two-dimensional torus induces some restrictions on adjacent homologies.

Lemma 3.3. *Assume $c \in \mathbb{E} \cap \mathbb{E}'$. If $(m, n) = g \in G_{m,\mathbb{E}}$ and $(m', n') = g' \in G_{m,\mathbb{E}'}$, then*

$$m'n - mn' = \pm 1.$$

Proof. By condition, $\tilde{\mathcal{A}}(c)$ contains homoclinic orbits with two types (m, n) and (m', n') . Clearly, $(m, n) = 1$ (m is prime to n) and $(m', n') = 1$. Since the configuration space is \mathbb{T}^2 , these curves intersect each other only at one point, guaranteed by the Lipschitz graph property. Thus, it is equivalent to the problem that, for each integer k , the solutions of the equation

$$mx + ny = k, \quad m'x + n'y = 0$$

are always integers. \square

Let us study the structure of Mather set as well as Mañé set for the cohomology class close to the flat \mathbb{F}_0 .

For each indivisible homology class $0 \neq g \in H_1(\mathbb{T}^2, \mathbb{Z})$, either there does not exist $\lambda > 0$ such that $\mathcal{L}_\beta(\lambda g) \in \partial^* \mathbb{F}_0$, or some $\lambda_0 > 0$ exists such that $\mathcal{L}_\beta(\lambda_0 g) \in \partial^* \mathbb{F}_0$.

In the first case, there exists at least one closed orbit $d\gamma_\lambda$ with the rotation vector λg for each $\lambda > 0$, which supports the minimal measure. Clearly, $\mathcal{L}_\beta(\lambda g)$ approaches to $\partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ as $\lambda \rightarrow 0^+$. Indeed, if $d(c, \mathcal{L}_\beta(\lambda g)) \rightarrow 0$ holds for $c \in \partial^* \mathbb{F}_0$, one has $g \notin \mathcal{L}_\beta^{-1}(c)$. As the speed of γ_λ approaches to zero only when it approaches to the fixed point, $d(d\gamma_\lambda, \tilde{\mathcal{M}}(c)) \geq d > 0$ holds. But it violates the Lipschitz property.

In typical case, $\mathcal{L}_\beta(\lambda g)$ is an interval if $\lambda > 0$. If $g \in G_{m, \mathbb{E}_i}$, then $\mathcal{L}_\beta(\lambda g)$ approaches to certain sub-flat \mathbb{E}_i . If $g = k_i g_i + k_j g_j$ with indivisible $(k_i, k_j) \in \mathbb{Z}_+^2$, $g_i \in G_{m, \mathbb{E}_i}$, $g_j \in G_{m, \mathbb{E}_j}$, \mathbb{E}_i and \mathbb{E}_j are two adjacent sub-flats, the interval will shrink to a vertex, the pinch point of \mathbb{E}_i and \mathbb{E}_j as $\lambda \rightarrow 0^+$. Let $\lambda \rightarrow 0^+$, we have a sequence of sets $\{d\gamma_\lambda\} = \{\cup_t (\gamma_\lambda(t), \dot{\gamma}_\lambda(t))\}$. By the upper-semi continuity of Mañé set in the cohomology class, its Kuratowski upper limit set is obviously in the Aubry set for certain $c \in \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$, thus, consists of minimal homoclinic orbits to the fixed point. As c approaches to the vertex, the Mather set approaches to a set of figure eight:

$$\mathcal{M}(c) \rightarrow \gamma_i * \gamma_j,$$

where $\gamma_l \subset \mathcal{A}(E_l)$ is a minimal homoclinic orbit such that $[\gamma_l] = g_l$ for $l = i, j$.

To be more clear, we study this phenomenon in the finite covering space $\bar{M} = \bar{k}_1 \mathbb{T} \times \bar{k}_2 \mathbb{T}$ where $\bar{k}_m = k_i g_{im} + k_j g_{jm}$ for $m = 1, 2$ if we write $g_i = (g_{i1}, g_{i2})$ and $g_j = (g_{j1}, g_{j2})$. Let σ be a permutation of $k_i + k_j$ symbols $(i, \dots, i, j, \dots, j)$, i.e. we have a map $\sigma: \{1, 2, \dots, k_i + k_j\} \rightarrow \{i, j\}$ such that the cardinality $\#\sigma^{-1}(i) = k_i$ and $\#\sigma^{-1}(j) = k_j$. The lift of homoclinic curve γ_i as well as γ_j to \bar{M} contains several curves, each of them is not closed curve. Pick up one curve $\bar{\gamma}_{\sigma(1)}$ in the lift of $\gamma_{\sigma(1)}$, it determines a unique curve $\bar{\gamma}_{\sigma(2)}$ such that the end point of $\bar{\gamma}_{\sigma(1)}$ is the starting point of $\bar{\gamma}_{\sigma(2)}$, and so on. Clearly, there exists certain permutation σ such that one Aubry class in $\mathcal{A}(c, \bar{M})$

$$\mathcal{A}_i(c, \bar{M}) \rightarrow \bar{\gamma}_{\sigma(1)} * \bar{\gamma}_{\sigma(2)} * \dots * \bar{\gamma}_{\sigma(k_i + k_j)}$$

as c approaches to the vertex along the path in the channel, see Figure 2 and 3.

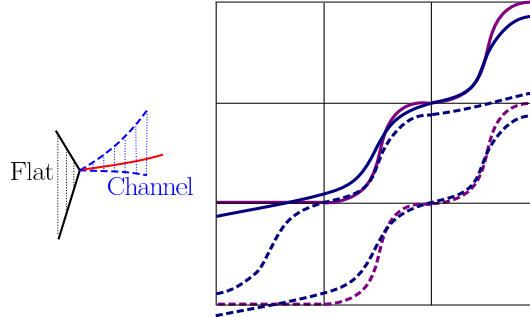


FIGURE 2. $[\gamma_1] = (1, 0)$, $[\gamma_2] = (1, 1)$, $k_1 = 1$, $k_2 = 2$. For each class c on the red line in the channel, $\rho(\mu_c) = \lambda([\gamma_1] + 2[\gamma_2])$. The solid blue line represent a periodic curve, the solid purple line represent the conjunction of the minimal homoclinic curves. The dashed lines represent the image of Deck-transformation.

In the second case, some $\lambda_0 > 0$ exists such that $\mathcal{L}_\beta(\lambda g) \in \partial^* \mathbb{F}_0$. In this case, certain sub-flat \mathbb{E}_i exists such that $\mathcal{L}_\beta(\lambda g) = \mathbb{E}_i \subset \partial^* \mathbb{F}_0$. In typical case, c -minimal

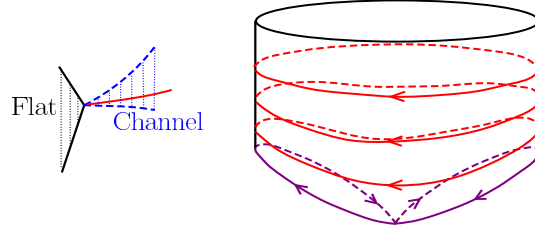


FIGURE 3. For each c on the red line in the channel, the Aubry set is a closed orbit in the cylinder, these closed orbits approaches to a curve of figure eight

measure is supported on a periodic orbit which is normally hyperbolic for each $c \in \mathbb{E}_i$. Thus, there is a channel C ended at \mathbb{E}_i such that $\cup_{c \in C} \tilde{\mathcal{M}}(c)$ makes up a normally hyperbolic cylinder.

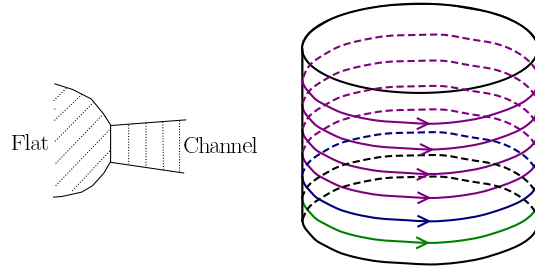


FIGURE 4. For each c in the channel, the Aubry set is a closed orbit in the cylinder (in purple), the closed circle in blue is in the Mather for c at the common boundary of the flat and the channel, the closed orbit in green is not local minimal orbit, but not global.

In this paper, we study the dynamics on certain energy level $H(x, y, y_3) = E$. Since one obtains the condition $\partial_{y_3} H \neq 0$ from the normal form, some function $y_3 = G(x, y)$ solves the equation $H = E$. Treat $G(x, y)$ as the new Hamiltonian and let $\tau = -x_3$ be the time, one obtains a system with two degrees of freedom, which is equivalent to the dynamics on the energy level $\{H^{-1}(E)\}$.

Theorem 3.4. *Assume the autonomous Hamiltonian $H(x, y, x_n, y_n)$ satisfies the condition $\partial_{y_n} H \neq 0$ on $\{H^{-1}(E)\} \cap \{y_n \in [y_n^-, y_n^+]\}$. Let $y_n = G(x, y, \tau)$ be the solution of $H = E$ ($\tau = -x_n$),*

$$L_H(x, x_n, \dot{x}, \dot{x}_n) = \max_{y, y_n} \langle (\dot{x}, \dot{x}_n), (y, y_n) \rangle - H(x, y, x_n, y_n),$$

$$L_G(x, y, \tau) = \max_y \langle \dot{x}, y \rangle - G(x, y, \tau),$$

α_H be the α -function for L_H and α_G be the α -function for L_G , then $(c, \alpha_G(c)) \in \alpha_H^{-1}(E)$ if $G(c) \in [y_n^-, y_n^+]$.

Proof. Let us keep in mind that $G(x, y, -x_n)$ solves the equation $H(x, y, x_n, y_n) = E$ and $\partial_{y_n} H \neq 0$. Let $\tilde{c} = (c, \alpha_G(c))$, $\tilde{\gamma} = (\gamma, \gamma_n)$, $\tilde{x} = (x, x_n)$ and $\tilde{y} = (y, y_n)$. Let γ be c -minimal curve for the Lagrange flow $\phi_{L_F}^t$, $\tilde{\gamma}$ is then \tilde{c} -minimal curve for the Lagrange flow $\phi_{L_H}^t$ if $\gamma_n = x_n$ and $\tilde{\gamma}$ is re-parameterized $\tau \rightarrow t$. Indeed, once $x = x(\tau)$

be a solution of $\phi_{L_F}^t$, one obtains $y = y(\tau)$ from the Hamiltonian equations. Be aware of the fact that $H(\tilde{x}(t), \tilde{y}(t)) \equiv E$, one find

$$\begin{aligned} [A_G(\gamma)] &= \int \left(\left\langle \frac{dx}{d\tau}, y - c \right\rangle - y_n + \alpha_G(c) \right) d\tau \\ &= \int (\langle \dot{\tilde{x}}, \tilde{y} - \tilde{c} \rangle - H + E) dt \\ &= [A_H(\tilde{\gamma})]. \end{aligned}$$

This completes the proof. \square

Let $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the standard projection $\pi_n \tilde{x} = x$. By this theorem, $\pi_n^{-1} : H^1(\mathbb{T}^{n-1}, \mathbb{R}) \rightarrow \alpha_H(E)$ is a homeomorphism when it is restricted in a neighborhood of the flat \mathbb{F}_0 of α_F . The pictures we obtained in this subsection remain same for $\tilde{c} \in \pi_n^{-1}(\mathbb{F}_0 + d) \cap \alpha_H^{-1}(E)$, where $\mathbb{F}_0 + d$ denotes d -neighborhood of \mathbb{F}_0 .

4. NORMALLY HYPERBOLIC INVARIANT CYLINDER

In this section, the existence normally hyperbolic invariant cylinder is investigated by using the normal form obtained in the section 3, where finitely many steps of KAM iteration and one step of linear coordinate transformation were carried out for obtaining the normal form. All these coordinate transformations are symplectic. Since Aubry set and Mañé set are symplectic invariants [Be2], it is good enough to study these objects by considering the normal form.

4.1. Homogenized Hamiltonian. The normal form of the Hamiltonian takes the form

$$(4.1) \quad H = h(\tilde{y}) + \epsilon Z(x, \tilde{y}) + \epsilon R(\tilde{x}, \tilde{y}),$$

where $\tilde{x} = (x, x_3) = (x_1, x_2, x_3)$ and $\tilde{y} = (y, y_3) = (y_1, y_2, y_3)$. As h is strictly convex, a unique curve exists along which $\partial_y h = (0, 0)$. This curve passes through the energy level $\{h^{-1}(E)\}$ transversally at unique point \tilde{y}_0 . As $E > \min \alpha$, one has $\partial_{y_3} h(\tilde{y}_0) = \omega_E \neq 0$. By translation, we assume $\tilde{y} = 0$.

As the dynamics we are concerned about is restricted on an energy level $\{H^{-1}(E)\}$, it can be reduced to a system with two and half degrees of freedom by standard method. Indeed, because of $\partial_{y_3} h(0) = \omega_E \neq 0$, the equation $H(\tilde{x}, \tilde{y}) = E$ uniquely determines a smooth function $y_3 = y_3(x, y, x_3)$ in certain neighborhood of $y_3 = 0$. Treating $-\omega y_3 = G$ as a new Hamiltonian and $\omega^{-1} x_3 = \tau$ as the new time variable, we obtain a time-periodic system with two degrees of freedom. From the normal form we find that

$$G(x, y, \tau) = h'(y) + \epsilon Z'(x, y) + \epsilon R'(x, y, \tau)$$

where $h'(0) = 0$, $\partial_y h'(0) = 0$ and $\epsilon R'$ is as small as ϵR .

As the first step, normally hyperbolic cylinder will be studied in the range of $\|y - y^*\| = O(\sqrt{\epsilon})$, one obtains the cylinder in $O(1)$ scale by choosing different y_0 . By rescaling variables $y - y^* = \sqrt{\epsilon} p$, $s = \sqrt{\epsilon} \tau$, one obtains an equivalent Hamiltonian

equation

$$\begin{aligned}\frac{dx}{ds} &= \frac{\omega^*}{\sqrt{\epsilon}} + Ap + \left(\frac{\partial h'(\sqrt{\epsilon}p)}{\epsilon \partial p} - \frac{\omega^*}{\sqrt{\epsilon}} - Ap \right) + \frac{\partial Z'}{\partial p} + \frac{\partial R'}{\partial p}, \\ \frac{dp}{ds} &= -\frac{\partial Z'}{\partial x} - \frac{\partial R'}{\partial x}\end{aligned}$$

which corresponds to the Hamiltonian

$$G_\epsilon = \frac{1}{\sqrt{\epsilon}} \langle \omega^*, p \rangle + \frac{1}{2} \langle A^* p, p \rangle + V^*(x) + Z'_\epsilon(x, \sqrt{\epsilon}p) + R'_\epsilon(x, \sqrt{\epsilon}p, s/\sqrt{\epsilon})$$

where $\omega^* = \partial h'(y^*)$, $A^* = \partial^2 h'(y^*)$, $V^*(x) = Z'(x, y^*)$ and

$$Z'_\epsilon = \frac{1}{\epsilon} h'(y^* + \sqrt{\epsilon}p) - \frac{1}{\sqrt{\epsilon}} \langle \omega^*, p \rangle - \langle A^* p, p \rangle + Z'(x, \sqrt{\epsilon}p + y^*) - Z'(x, y^*).$$

Clearly, one has $Z'_\epsilon = O(\sqrt{\epsilon})$ and $\|R'_\epsilon\|_{C^2} = O(\epsilon^{(r-2)\sigma})$, where the C^2 -norm is with respect to (x, p) only. Obviously, one has

Proposition 4.1. *Each orbit $(x(s), p(s))$ of the Hamiltonian flow $\Phi_{G_\epsilon}^s$ uniquely determines an orbit $(x(\tau), y(\tau)) = (x(s/\sqrt{\epsilon}), y^* + \sqrt{\epsilon}p(s/\sqrt{\epsilon}))$ of Φ_G^τ . If $G_\epsilon(x(s), p(s)) = E_\epsilon$ and $G(x(\tau), y(\tau)) = E$, then $E = \epsilon E_\epsilon$.*

The Hamiltonian G_ϵ is a small perturbation of the homogenized Hamiltonian

$$\bar{G} = \frac{1}{\sqrt{\epsilon}} \langle \omega^*, p \rangle + \frac{1}{2} \langle A^* p, p \rangle + V^*(x).$$

It is uniquely related to the Lagrangian

$$\bar{L} = -\frac{1}{\sqrt{\epsilon}} \langle \omega^*, A^{*-1} \dot{x} \rangle + \frac{1}{2} \langle A^{*-1} \dot{x}, \dot{x} \rangle - V^*(x).$$

Let us first consider the case when $y^* = 0$, it follows that $\omega^* = 0$. Correspondingly, let $A = A^*$ and $V = V^*$ for $y^* = 0$:

$$\bar{G} = \frac{1}{2} \langle Ap, p \rangle + V(x), \quad \bar{L} = \frac{1}{2} \langle A^{-1} \dot{x}, \dot{x} \rangle - V(x).$$

For this Hamiltonian system, the maximal point of V determines a stationary solution which corresponds to a minimal measure of \bar{L} , where the matrix

$$\begin{pmatrix} 0 & A \\ -\partial_x^2 V & 0 \end{pmatrix}$$

has 4 real eigenvalue $\pm \lambda_1, \pm \lambda_2$. By translation of coordinates, it is a generic condition that

(H1): *V attains its maximum at $x = 0$ only, the Hessian matrix of V at $x = 0$ is negative definite. All eigenvalues are different: $-\lambda_2 < -\lambda_1 < 0 < \lambda_1 < \lambda_2$.*

Such a hypothesis leads to certain hyperbolicity of minimal homoclinic orbits. Let us consider the case: for $c \in \partial \mathbb{F}_0$ the Aubry set $\mathcal{A}(c) = \cup_{t \in \mathbb{R}} \zeta(t)$, where $\zeta: \mathbb{R} \rightarrow M$ is a minimal homoclinic curve. By the assumption **H1**, the fixed point $z = (x, y) = 0$ has its locally stable manifold W^+ as well as the locally unstable manifold W^- . They intersect each other transversally at the origin. As each homoclinic orbit entirely stays in the stable as well as in the unstable manifolds, along such orbit their intersection can not be transversal in the standard definition. However, for convenience and without of confusion, we call the intersection transversal still if

$$T_x W^- \oplus T_x W^+ = T_x H^{-1}(E), \quad \forall x \in H^{-1}(E).$$

If we denote by $\Lambda_i^+ = (\Lambda_{xi}, \Lambda_{yi})$ the eigenvector corresponding to the eigenvalue λ_i , then the eigenvector for $-\lambda_i$ will be $\Lambda_i^- = (\Lambda_{xi}, -\Lambda_{yi})$. It is a generic condition also that

(H2): *for each $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, there is at most one minimal orbit associated with the class g , the stable manifold intersects the unstable manifold transversally along each minimal homoclinic orbit. Each minimal homoclinic orbit approaches to the fixed point along the direction $\Lambda_1: \dot{\gamma}(t)/\|\dot{\gamma}(t)\| \rightarrow \Lambda_{x1}$ as $t \rightarrow \pm\infty$.*

Because of the separability property, the countability of homology classes of all homoclinic curves and Theorem 3.3, we have another generic condition

(H3): *For any $c \in \partial^*\mathbb{F}_0$, the Aubry set does not contain minimal curve homoclinic to the origin (fixed point).*

4.2. Cylinder for truncated Hamiltonian: near double resonance. Let us start with a Hamiltonian with two and half degrees of freedom:

$$(4.2) \quad G_\epsilon = \frac{1}{2}\langle Ap, p \rangle + V(x) + Z'_\epsilon(x, p) + R'_\epsilon(x, \sqrt{\epsilon}p, s/\sqrt{\epsilon})$$

where $(x, \sqrt{\epsilon}p, s/\sqrt{\epsilon}) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{T}$, $Z'' = O(\sqrt{\epsilon})$, $\|\epsilon R'\|_{C^2} = O(\epsilon^{1+(r-2)\sigma})$ where the C^2 -norm is with respect to (x, p) only. Recall the homogenized Hamiltonian as well as the homogenized Lagrangian

$$\bar{G}(x, p) = \frac{1}{2}\langle Ap, p \rangle + V(x), \quad \bar{L}(x, \dot{x}) = \frac{1}{2}\langle A^{-1}\dot{x}, \dot{x} \rangle - V(x)$$

where $\dot{x} = \frac{dx}{ds}$.

By the assumption **(H1)**, the origin $(x, \dot{x}) = 0$ is a fixed point which supports a c -minimal measure for each $c \in \mathbb{F}_0$, where $\mathbb{F}_0 \in H^1(\mathbb{T}^2, \mathbb{R})$ is a 2-dimensional flat. As classified before, for each $c \in \partial\mathbb{F}_0 \setminus \partial^*\mathbb{F}_0$ the Aubry set consists of minimal homoclinic orbits plus the fixed point. While for $c \in \partial^*\mathbb{F}_0$, besides the minimal measure supported the fixed point, there exists another c -minimal measure.

Let us consider this issue from homological point of view. Given an irreducible class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, it is not necessary that some minimal homoclinic curve γ exists such that $[\gamma] = g$. However, for each $\nu > 0$, there exists a periodic curve $\gamma_\nu: [0, T_\nu] \rightarrow \mathbb{T}^2$ minimizing the action along curves with the same period

$$A_{\bar{G}}(\gamma_\nu) = \inf_{\substack{[\xi] = g \\ \xi(0) = \xi(T_\nu)}} \int_0^{T_\nu} L_{\bar{G}}(\xi(t), \dot{\xi}(t)) dt,$$

and its rotation vector $\rho(\gamma_\nu, \dot{\gamma}_\nu) = \nu g$. As the configuration space is a 2-dimensional torus, this periodic orbit supports a c -minimal measure for $c \in \mathcal{L}_\beta(\nu g)$. Denote by (x_ν, y_ν) the periodic orbit determined by γ_ν in the phase space, where $x_\nu = \gamma_\nu$.

For each finite period T_ν , it is generic that the minimal measure is supported on at most two periodic orbits, both are hyperbolic, namely, it has the stable and unstable manifold. Once g and ν are given, a unique energy $E = E(g, \nu)$ exists such that the periodic orbits stay in the energy level $\bar{G}^{-1}(E)$. These periodic orbits constitute a two-dimensional cylinder. However, it appears not reasonable to assume the normal hyperbolicity for $\Phi_{\bar{G}}^s$ in usual sense as the speed along the orbit may undergo large variation, especially, when it is very close to some homoclinic orbit. Therefore, one

can not see the separation of the spectrum of $D\Phi_{\bar{G}}^s$ in normal and in tangent direction. As the first step, let us study the case when $c \rightarrow \partial\mathbb{F}_0$, i.e. decrease the parameter ν .

There are two cases alternatively as ν decreases: $\mathcal{L}_\beta(\nu g) \rightarrow \partial\mathbb{F}_0$ as ν decrease to zero, or $\exists \nu_0 > 0$ such that $\mathcal{L}_\beta(\nu_0 g) \in \partial\mathbb{F}_0$.

The second case is easy to handle. Since the class g is irreducible, there is some periodic orbit $(x_{\nu_0}, \dot{x}_{\nu_0})$ which minimizes the c -action provided $c \in \mathcal{L}_\beta(\nu_0 g)$. In this case, the normally hyperbolicity of the cylinder is obvious provided this periodic orbit is non-degenerate in the sense of Morse.

In the first case, the cohomology class approaches to some sub-flat $\mathbb{E}_i \subset \partial\mathbb{F}_0 \setminus \partial^*\mathbb{F}_0$ or to some vertex where two adjacent sub-flats $\mathbb{E}_i, \mathbb{E}_{i+1} \subset \partial\mathbb{F}_0 \setminus \partial^*\mathbb{F}_0$ joint together. Under the hypothesis **(H2)**, the Aubry set $\mathcal{A}(c)$ contains exactly one minimal homoclinic curve and the point of the origin provided the class c is in the interior of some edge (sub-flat). Let γ_j be the minimal homoclinic curve related to \mathbb{E}_j for $j = i, i+1$, it determines the minimal homoclinic orbit $(x_i(s), p_i(s)) \subset \bar{G}^{-1}(0)$. Denote by $g_j = [\gamma_j]$ the homology class. If $g = g_i$, $\gamma_\nu \rightarrow \gamma_i$ as ν decreases to zero. If there exist two positive integers k_i, k_{i+1} such that $g = k_i g_i + k_{i+1} g_{i+1}$, the curve x_ν approaches to the set $\cup_{t \in \mathbb{R}} x_i(t) \cup x_{i+1}(t)$ (figure eight) as $\nu \rightarrow 0$, folding k_i and k_{i+1} -times along x_i and x_{i+1} respectively. Note that all these three classes are irreducible.

Each $\nu > 0$ determines the energy $E > 0$ such that $(x_\nu(s), p_\nu(s)) \subset \bar{G}^{-1}(E)$. By the study in Section 4.2, (Eq. (3.10)), for $g = k_i g_i + k_{i+1} g_{i+1}$ and suitably small $E > 0$

$$T_\nu = T_\nu(E, g) = \tau_{E, g} - \frac{1}{\lambda_1} (k_i + k_{i+1}) \ln E$$

where $\tau_{E, g} \rightarrow k_i \tau_{E, g_i} + k_{i+1} \tau_{E, g_{i+1}}$ as $E \rightarrow 0$, both τ_{E, g_i} and $\tau_{E, g_{i+1}}$ is bounded as $E \rightarrow 0$.

To study the dynamics around the minimal homoclinic orbits $z_\ell = (x_\ell, p_\ell)$ ($\ell = i, i+1$), we use a new canonical coordinates (x, p) such that, restricted in a small neighborhood of $z = 0$, one has

$$\bar{G} = \frac{1}{2}(p_1^2 - \lambda_1^2 x_1^2) + \frac{1}{2}(p_2^2 - \lambda_2^2 x_2^2) + P_3(x)$$

where $P_3(x) = O(\|x\|^3)$. Without losing generality, we assume $x_{\ell,1}(s) \downarrow 0$ as $s \rightarrow -\infty$, $x_{\ell,1}(s) \uparrow 0$ as $s \rightarrow \infty$ and $\dot{x}_\ell(s)/\|\dot{x}_\ell(s)\| \rightarrow (1, 0)$ as $s \rightarrow \pm\infty$. Here the notation is taken as granted: $x_\ell = (x_{\ell,1}, x_{\ell,2})$. We choose 2-dimensional disk lying in the energy level set $\bar{G}^{-1}(E)$

$$\Sigma_{E, \delta}^\mp = \{(x, p) \in \mathbb{R}^4 : \|(x, p)\| \leq d, \bar{G}(x, p) = E, x_1 = \pm\delta\}.$$

Because of the special form of \bar{G} , one has

$$\Sigma_{0, \delta}^\mp = \{x_1 = \pm\delta, p_1^2 + p_2^2 - \lambda_2^2 x_2^2 = \lambda_1^2 \delta^2 - 2P_3(\pm\delta, x_2), \|(x, p)\| \leq d\}.$$

Let W^- (W^+) denote the unstable (stable) manifold of the fixed point which entirely stays in the energy level set $\bar{G}^{-1}(0)$. The tangent vector of $W^- \cap \Sigma_{0, \delta}^-$ takes the form

$$v_\delta^- = (v_{x_1}, v_{x_2}, v_{p_1}, v_{p_2}) = (0, \pm 1, p_{1, \delta}, \pm \lambda_2 + p_{2, \delta}) \in T_{z_\delta^-}(W^- \cap \Sigma_{0, \delta}^-)$$

where both $p_{1, \delta}$ and $p_{2, \delta}$ are small number.

Denote by $T_{\delta,\ell}^\pm$ the time when the homoclinic orbit $z_\ell(s)$ passes through $\Sigma_{0,\delta}^\pm$. As $\partial_{y_1}\bar{G} > 0$ at the point on the orbit where $x_1 = \pm\delta$ and both homoclinic orbits $z_i(s)$ and $z_{i+1}(s)$ approach in the same direction to the fixed point, both $\Sigma_{0,\delta}^+$ and $\Sigma_{0,\delta}^-$ intersect these two homoclinic orbits transversally. Let $z_{\delta,\ell}^\pm$ denote the intersection point of $z_\ell(s)$ with $\Sigma_{0,\delta}^\pm$. In a small neighborhood of that point $B_\varepsilon(z_{\delta,\ell}^-)$, one obtains a map $\Psi_{0,\delta}: \Sigma_{0,\delta}^- \cap B_\varepsilon(z_{\delta,\ell}^-) \rightarrow \Sigma_{0,\delta}^+$ in following way, starting from a point z in this neighborhood, there is a unique orbit which moves along $z_\ell(s)$ and comes to a point $\Psi_{0,\delta}(z) \in \Sigma_{0,\delta}^+$ after a time approximately equal to $T_{\delta,\ell}^+ - T_{\delta,\ell}^-$.

Let us fix small D . There exists $a_1 > 1$ (depending on D) such that

$$a_1^{-1} \leq \|D\Psi_{0,D}(z_{D,j}^-)|_{T(W^- \cap \Sigma_{0,D}^-)}\|, \|D\Psi_{0,D}^{-1}(z_{D,j}^+)|_{T(W^+ \cap \Sigma_{0,D}^+)}\| \leq a_1.$$

One can choose larger a_1 so that this estimate works for both $\ell = i$ and $\ell = i + 1$. Clearly, one has $a_1 \rightarrow \infty$ as $D \rightarrow 0$.

As the homoclinic curves approach to the origin in the direction of $(1, 0)$, for small $\delta \ll D$, there exists a constant $\mu_1 > 0$ such that $\mu_1 \downarrow 0$ as $D \rightarrow 0$ and

$$\frac{1}{\lambda_1 + \mu_1} \ln\left(\frac{D}{\delta}\right) \leq T_{D,\ell}^\pm - T_{\delta,\ell}^\pm \leq \frac{1}{\lambda_1 - \mu_1} \ln\left(\frac{D}{\delta}\right).$$

The Hamiltonian flow $\Phi_{\bar{G}}^t$ defines a map $\Psi_{0,\delta,D}^-: \Sigma_{0,\delta}^- \rightarrow \Sigma_{0,D}^-$. Emanating from a point in $\Sigma_{0,\delta}^-$ there exists a unique orbit which arrives $\Sigma_{0,D}^-$ after a time bounded by the last formula. The flow $\Phi_{\bar{G}}^t$ also defines a map $\Psi_{0,\delta,D}^+: \Sigma_{0,\delta}^+ \rightarrow \Sigma_{0,D}^+$. Starting from a point in $\Sigma_{0,D}^+$ an orbit intersects $\Sigma_{0,\delta}^+$ after a time also bounded by the last formula.

Restricted in the ball B_D , let us consider the variational equation of the flow $\Psi_{\bar{G}}^s$ along the homoclinic orbit $z_j(s)$. It follows from the normal form of the homogenized Hamiltonian \bar{G} that the tangent vector $(\Delta x, \Delta p) = (\Delta x_1, \Delta x_2, \Delta p_1, \Delta p_2)$ satisfies the variational equation

$$(4.3) \quad \Delta \dot{x}_i = \Delta p_i, \quad \Delta \dot{p}_i = \lambda_i^2 \Delta x_i - \Psi_{1i}(x_\ell(s)) \Delta x_1 - \Psi_{2i}(x_\ell(s)) \Delta x_2$$

for $i = 1, 2$, where $\Psi_{ij} = \partial_{x_i} \partial_{x_j} P_3$. Some positive constant $a > 0$ exists such that $|\Psi_{ij}(x_\ell(s))| \leq a \|x_\ell(s)\|$ provided $\|x_\ell(s)\|$ is small. Since the homoclinic orbit approaches to the fixed point in the direction of $(\dot{x}, \dot{p}) = (1, 0, \lambda_1^2, 0)$,

$$De^{-(\lambda_1 + \mu_1)(s - T_{D,\ell}^+)} \leq \|x(s)\|_{[T_{D,\ell}^+, \infty)} \leq De^{-(\lambda_1 - \mu_1)(s - T_{D,\ell}^+)}.$$

Thus, for the initial value $\Delta z(T_{D,\ell}^+) = (\Delta x(T_{D,\ell}^+), \Delta p(T_{D,\ell}^+))$ satisfying the condition

$$|\langle \Delta z(T_{D,\ell}^+), v_\delta^- \rangle| \geq 2/3 \|\Delta z(T_{D,\ell}^+)\| \|v_\delta^-\|$$

($v_\delta^- = (0, \pm 1, p_{1,\delta}, \pm \lambda_2 + p_{2,\delta})$) one obtains from the hyperbolicity that

$$a_2^{-1} \|\Delta z(T_{D,\ell}^+)\| e^{(\lambda_2 - \mu_1)(T_{\delta,\ell}^+ - T_{D,\ell}^+)} \leq \|\Delta z(T_{\delta,\ell}^+)\| \leq a_2 \|\Delta z(T_{D,\ell}^+)\| e^{(\lambda_2 + \mu_1)(T_{\delta,\ell}^+ - T_{D,\ell}^+)}$$

holds for some positive constant $a_2 > 1$ depending on λ_i as well as on P . Therefore, for each vector $v \in T_{z_D^+} \Sigma_{0,D}^+$ which is nearly parallel to $T_{z_D^+}(W^- \cap \Sigma_{0,D}^+)$: $|\langle v, v' \rangle| \geq \frac{2}{3} \|v\| \|v'\|$ holds for $v' \in T_{z_D^+}(W^- \cap \Sigma_{0,D}^+)$ we have

$$a_2^{-1} \left(\frac{D}{\delta}\right)^{\frac{\lambda_2}{\lambda_1} - \mu_2} \leq \lim_{\|v\| \rightarrow 0} \frac{\|D\Psi_{0,\delta,D}^+(z_{D,\ell}^+)v\|}{\|v\|} \leq a_2 \left(\frac{D}{\delta}\right)^{\frac{\lambda_2}{\lambda_1} + \mu_2}.$$

Similarly, one has

$$a_3^{-1} \left(\frac{D}{\delta} \right)^{\frac{\lambda_2}{\lambda_1} - \mu_2} \leq \|D\Psi_{0,\delta,D}^-(z_{\delta,\ell}^-)\|_{T_{z_{\delta,\ell}^-}(W^- \cap \Sigma_{0,\delta}^-)} \leq a_3 \left(\frac{D}{\delta} \right)^{\frac{\lambda_2}{\lambda_1} + \mu_2},$$

where $a_3 > 1$ also depends on λ_i as well as on P , $\mu_2 > 0$ and $\mu_2 \rightarrow 0$ as $D \rightarrow 0$.

By construction, the intersection of the 2-dimensional disk $\Sigma_{0,\delta}^-$ with the unstable manifold W^- constitutes a curve. Let $\Gamma_{\delta,\ell}^- \subset W^- \cap \Sigma_{0,\delta}^-$ be a very short segment of this curve, passing through the point $z_{\delta,\ell}^-$ where the homoclinic orbit z_ℓ intersects $\Sigma_{0,\delta}^\pm$. Pick up a point z_ℓ^* on the homoclinic orbit z_ℓ far away from the fixed point and take a 2-dimensional disk $\Sigma_\ell^* \subset \bar{G}^{-1}(0)$ passing through this point which is transversal to the flow Φ_G^s in the sense that $T_{z_\ell^*} \bar{G}^{-1}(0) = \text{span}(T_{z_\ell^*} \Sigma_\ell^*, J\nabla \bar{G}(z_\ell^*))$. The Hamiltonian flow Φ_G^s sends each point of $\Gamma_{\delta,\ell}^-$ to this disk provided it is close to z_ℓ^- . In this way, one obtains a map $\Psi_{\delta,\ell}^{-,*}: \Sigma_{0,\delta}^- \rightarrow \Sigma_\ell^*$. Let $\Gamma_{\delta,\ell}^{-,*} = \Psi_{\delta,\ell}^{-,*} \Gamma_{\delta,\ell}^-$. According to the assumption **(H2)**, one has $T_{z_\ell^*} \bar{G}^{-1}(0) = \text{span}(T_{z_\ell^*} W^+, T_{z_\ell^*} W^-)$. Thus, one also has $T_{z_\ell^*} \bar{G}^{-1}(0) = \text{span}(T_{z_\ell^*} W^+, T_{z_\ell^*} \Gamma_{\delta,\ell}^{-,*})$. Consequently, it follows from the λ -lemma that $\Psi_{0,\delta}(\Gamma_{\delta,\ell}^-)$ keeps C^1 -close to $W^- \cap \Sigma_{0,\delta}^+$ at the point $z_{\delta,\ell}^+$ and $\Psi_{0,\delta}^{-1}(\Gamma_{\delta,\ell}^+)$ keeps C^1 -close to $W^+ \cap \Sigma_{0,\delta}^-$ at the point $z_{\delta,\ell}^-$ provided $\delta > 0$ is sufficiently small. As $\Psi_{0,\delta} = \Psi_{0,\delta,D}^- \circ \Psi_{0,D}^+ \circ \Psi_{0,\delta,D}^+$, one obtains

$$a_4^{-1} \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} - \mu_2)} \leq \|D\Psi_{0,\delta}(z_{\delta,\ell}^-)\|_{T_{z_{\delta,\ell}^-}(W^- \cap \Sigma_{0,\delta}^-)} \leq a_4 \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} + \mu_2)},$$

and

$$a_4^{-1} \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} - \mu_2)} \leq \|D\Psi_{0,\delta}^{-1}(z_{\delta,\ell}^+)\|_{T_{z_{\delta,\ell}^+}(W^+ \cap \Sigma_{0,\delta}^+)} \leq a_4 \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} + \mu_2)},$$

where $a_4 = a_1 a_2 a_3 > 1$. See Figure 5.

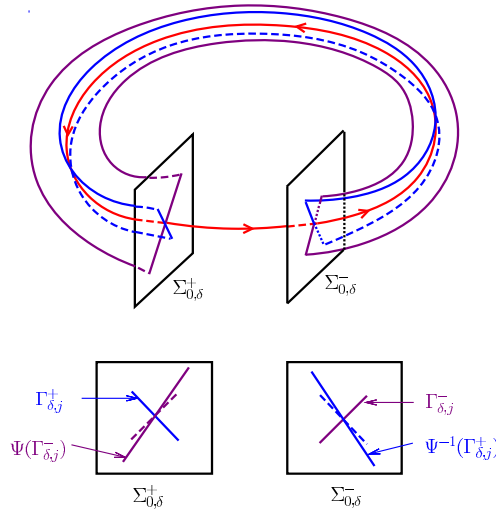


FIGURE 5.

For $E > 0$ sufficiently small, $\Sigma_{E,\delta}^\pm$ is C^{r-1} -close to $\Sigma_{0,\delta}^\pm$ respectively. Let $z_E(s) = (x_E(s), p_E(s))$ be the minimal periodic orbit staying in the energy level set $\bar{G}^{-1}(E)$, it approaches the homoclinic orbit as E decreases to zero. Thus, it passes through

the section $\Sigma_{E,\delta}^-$ as well as $\Sigma_{E,\delta}^+$ $k_1 + k_2$ times for one period provided $E > 0$ is sufficiently small. We number these points as $z_{E,k}^\pm$ ($k = 1, 2, \dots, k_1 + k_2$) by the role that emanating from a point $z_{E,k}^-$, the orbit reaches to the point $z_{E,k+1}^+$ first and then to the point $z_{E,k+2}^-$ and so on. Restricted on small neighborhoods of these points, the flow Φ_G^t defines a local diffeomorphism $\Psi_{E,\delta}: \Sigma_{E,\delta}^- \rightarrow \Sigma_{E,\delta}^+$. Because of the smooth dependence of ODE solutions on initial data, a small $\varepsilon > 0$ exists such that, for the vector v^\pm ε -parallel to $T_{z_\delta^\pm}(W^\pm \cap \Sigma_{0,\delta}^\pm)$ in the sense that $|\langle v^\pm, v_0^\pm \rangle| \geq (1 - \varepsilon)\|v^\pm\|\|v_0^\pm\|$ holds for some $v_0^\pm \in T_{z_\delta^\pm}(W^\pm \cap \Sigma_{0,\delta}^\pm)$, we obtain from the hyperbolicity of $\Psi_{0,\delta}$ (see the formulae above Figure 5) that

$$a_5^{-1} \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} - \mu_3)} \leq \frac{\|D\Psi_{E,\delta}(z_{E,k}^-)v^-\|}{\|v^-\|} \leq a_5 \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} + \mu_3)},$$

and

$$a_5^{-1} \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} - \mu_3)} \leq \frac{\|D\Psi_{E,\delta}^{-1}(z_{E,k}^+)v^+\|}{\|v^+\|} \leq a_5 \left(\frac{D}{\delta} \right)^{2(\frac{\lambda_2}{\lambda_1} + \mu_3)},$$

holds each j , where $a_5 > a_4 > 1$, $0 < \mu_3 \rightarrow 0$ as $D \rightarrow 0$. Once the vector v^- is chosen ε -parallel to $T_{z_\delta^-}(W^- \cap \Sigma_{0,\delta}^-)$ then the vector $D\Psi_{E,\delta}(z_{E,k}^-)v^-$ is ε -parallel to $T_{z_\delta^+}(W^+ \cap \Sigma_{0,\delta}^+)$.

As $E > 0$, the Hamiltonian flow Φ_G^t defines naturally a diffeomorphism $\Psi_{E,\delta}^+: \Sigma_{E,\delta}^+ \rightarrow \Sigma_{E,\delta}^-$. According to the study in Section 4.2 (cf. formula (3.10)), starting from $\Sigma_{E,\delta}^+$, the periodic orbit comes to $\Sigma_{E,\delta}^-$ after a time approximately equal to $\lambda_1^{-1}(|\ln E| - 2|\ln \delta|) + \tau_\delta$ in which τ_δ is uniformly bounded as $\delta \rightarrow 0$. Therefore, for a vector v^+ ε -parallel to $T_{z_{0,\delta}^+}(W^+ \cap \Sigma_{0,\delta}^+)$, one obtains that the vector $D\Psi_{E,\delta}^+(z_{E,j}^+)v^+$ is ε -parallel to $T_{z_{0,\delta}^-}(W^- \cap \Sigma_{0,\delta}^-)$ and

$$a_6^{-1} \left(\frac{\delta^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} - \mu_4} \leq \frac{\|D\Psi_{E,\delta,\delta}^+(z_{E,k}^+)v^+\|}{\|v^+\|} \leq a_6 \left(\frac{\delta^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} + \mu_4}$$

where $a_6 > 1$ for each k , $0 < \mu_4 \rightarrow 0$ as $\delta \rightarrow 0$. Similarly, for a vector v^- ε -parallel to $T_{z_{0,\delta}^-}(W^- \cap \Sigma_{0,\delta}^-)$, one obtains that the vector $D\Psi_{E,\delta,\delta}^-(z_{E,j}^-)v^-$ is ε -parallel to $T_{z_{0,\delta}^+}(W^+ \cap \Sigma_{0,\delta}^+)$ and

$$a_6^{-1} \left(\frac{\delta^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} - \mu_4} \leq \frac{\|D\Psi_{E,\delta,\delta}^{-1}(z_{E,k}^-)v^-\|}{\|v^-\|} \leq a_6 \left(\frac{\delta^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} + \mu_4}.$$

The composition of the two maps constitutes a Poincaré map $\Phi_{E,\delta} = \Psi_{E,\delta,\delta}^- \circ \Psi_{E,\delta}^+$, it maps a small neighborhood of the point $z_{E,k}^-$ in $\Sigma_{E,\delta}^-$ to a small neighborhood of the point $z_{E,k+1}^-$ in $\Sigma_{E,\delta}^-$. For a vector v^- ε -parallel to $T_{z_{0,\delta}^-}(W^- \cap \Sigma_{0,\delta}^-)$ the vector $D\Phi_{E,\delta}(z_{E,k}^-)v^-$ is still ε -parallel to $T_{z_{0,\delta}^-}(W^- \cap \Sigma_{0,\delta}^-)$

$$(4.4) \quad \Lambda^{-1} \left(\frac{D^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} - \mu_5} \leq \frac{\|D\Phi_{E,\delta}(z_{E,k}^-)v^-\|}{\|v^-\|} \leq \Lambda \left(\frac{D^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} + \mu_5},$$

and for a vector v^+ ε -parallel to $T_{z_{0,\delta}}^-(W^+ \cap \Sigma_{0,\delta}^-)$ the vector $D\Phi_{E,\delta}^{-1}(z_{E,k}^-)v^+$ is still ε -parallel to $T_{z_{0,\delta}}^-(W^+ \cap \Sigma_{0,\delta}^-)$

$$(4.5) \quad \Lambda^{-1} \left(\frac{D^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} - \mu_5} \leq \frac{\|D\Phi_{E,\delta}^{-1}(z_{E,k}^-)v^+\|}{\|v^+\|} \leq \Lambda \left(\frac{D^2}{E} \right)^{\frac{\lambda_2}{\lambda_1} + \mu_5}$$

holds for each k , where $\Lambda \geq a_5 a_6 > 1$, $0 < \mu_5 \rightarrow 0$ as $D \rightarrow 0$. Consequently, each point $z_{E,j}^-$ is a hyperbolic fixed point for the map $\Phi_{E,\delta}^{k_i + k_{i+1}}$, it is also hyperbolic for $\Phi_{E,\delta}$. Thus, the minimal periodic orbit z_E has its stable as well as unstable manifold, denote by W_E^+ and W_E^- respectively.

Lemma 4.1. *The conditions (H1) and (H2) are assumed. For a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, if $\mathcal{L}_\beta(\nu g) \rightarrow \partial \mathbb{F}_0$ as ν decrease to zero, then there exists $E_0 > 0$ such that for each $c \in \mathcal{L}_\beta(\nu g)$ with $\alpha(c) = E \in (0, E_0]$ the c -minimal measure μ_c is uniquely supported on a periodic orbit. Let $\Sigma_E \subset \bar{G}^{-1}(E)$ be a two-dimensional disk such that $T_z \bar{G}^{-1}(E) = \text{span}(T_z \Sigma_E, J \nabla \bar{G}(z))$ for $z \in \Sigma_E$ and let $\Phi_E: \Sigma_E \rightarrow \Sigma_E$ be the return map naturally determined by the flow Φ_G^t , there exists some $\lambda > 1, C > 0$ independent of $E \leq E_0$ such that*

$$\|D\Phi_E(z_{E,0})v^-\| \geq CE^{-\lambda}\|v^-\|, \quad \forall v^- \in T_{z_{E,0}}W_E^-;$$

$$\|D\Phi_E(z_{E,0})v^+\| \leq C^{-1}E^\lambda\|v^+\|, \quad \forall v^+ \in T_{z_{E,0}}W_E^+,$$

where $z_{E,0}$ is the point where the periodic orbit intersects Σ_E .

Proof. In this case, the cohomology class approaches to some sub-flat $\mathbb{E}_i \subset \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ or to some vertex where two adjacent sub-flats $\mathbb{E}_i, \mathbb{E}_{i+1} \subset \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ joint together. By the study above, one has such hyperbolicity shown in the inequality (4.4) and (4.5). By implicit function theorem, the uniqueness of periodic orbit for sufficiently small ν follows from this hyperbolicity. \square

In the second case, there exists some $\nu_0 > 0$ such that $\mathcal{L}_\beta(\nu_0 g) \in \partial^* \mathbb{F}_0$. It is typical that $\mathcal{L}_\beta(\nu_0 g)$ is a sub-flat $\mathbb{E}_i \subset \partial^* \mathbb{F}_0$, where the c -minimal measure is uniquely supported on a hyperbolic periodic orbit $z_0(s) \subset \bar{G}^{-1}(0)$. Thus, in the energy level set $\bar{G}^{-1}(0)$ suitable two-dimensional disk Σ_0 is chosen so that the Poncaré return map is hyperbolic at the fixed point. The uniqueness of minimal periodic orbit for ν close to ν_0 follows from the implicit function theorem. Therefore, the following condition is also generic

(H4): *Given a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, there are finitely many $\nu_i \in (0, \nu^*]$ such that, for each rotation vector $\nu_i g$, the Mather set consists of two periodic orbits, for any other rotation vector νg , the Mather set consists of exactly one periodic orbit. All these periodic orbits are hyperbolic.*

We call these $\{\nu_i\}$ bifurcation points. Let $\nu_1 > 0$ be the smallest one. Since there is correspondence $E = E(\nu, g)$ we introduce a notation

$$\Pi_{E,g} = \{(x_{E'}(s), p_{E'}(s)) : [x_{E'}] = g, E' \in [E, E(\nu_1, g)], s \in \mathbb{R}\}.$$

where $E > 0$ is small. It is a two-dimensional cylinder which consists of a family of periodic orbits, approaching to a curve of figure-of-eight as E decreases to zero. Let

$$\Pi_{E_0, E_1, g} = \{(x_{E'}(s), p_{E'}(s)) : [x_{E'}] = g, E' \in [E_0, E_1], s \in \mathbb{R}\} \subset \Pi_{E,g},$$

it is invariant for the Hamiltonian flow $\Phi_{\bar{G}}^t$ of \bar{G} . Let $T(E)$ denote the period of the periodic orbit in $\bar{G}^{-1}(E)$, one has

$$\int_{\Pi_{E_0, E_1, g}} \omega = \int_{E_0}^{E_1} \int_0^{T(E)} dE \wedge dt > 0.$$

The cylinder is not standard, but slant and crumpled. To see how the symplectic area is related to the usual area of the cylinder, let us study the dependence of the fixed point $(x_2(E), p_2(E))$ of the Poincaré return map $\Phi_{E, \delta}$ on E . By definition, the fixed point is a solution of the equation

$$(4.6) \quad \Phi_{E, \delta}(x_2(E), p_2(E)) - (x_2(E), p_2(E)) = 0.$$

Emanating from a point $(\delta, p_1, x_2, p_2) \in \bar{G}^{-1}(E)$ the orbit reach a point $z \in \{x_1 = -\delta\}$ after a time $\tau(E, \delta)$ which remains bounded as $E \downarrow 0$. Let $z' \in \{x_1 = -\delta\}$ be the point corresponding to $(\delta, p'_1, x_2, p_2) \in \bar{G}^{-1}(E')$, obtained in the same way. The difference of the (x_2, y_2) -coordinate of these two points is bounded by $d_0|p_1 - p'_1|$ where d_0 depends on δ . Let $(\Delta x, \Delta y)$ be the solution of the variational equation (4.3) along the periodic solution $(x_E(s), p_E(s))$ through a neighborhood of the origin ($|x_1| \leq \delta$), one obtains from the estimate (3.10) that

$$\|(\Delta x, \Delta y)(s_1)\| \leq d_1 E^{-\frac{\lambda_2}{\lambda_1} - \mu_6} \|(\Delta x, \Delta y)(s_0)\|$$

where the time s_0 and s_1 are chosen so that the first coordinate $x_{E,1}(s_0) = -\delta$ and $x_{E,1}(s_1) = \delta$. Therefore, one finds that

$$\left\| \frac{\partial \Phi_E}{\partial p_1} \right\| \leq d_0 d_1 E^{-\frac{\lambda_2}{\lambda_1} - \mu_6}$$

where $0 < \mu_6 \rightarrow 0$ as $\delta \rightarrow 0$. As the quantity $\left\| \frac{\partial \Phi_E}{\partial (x_2, p_2)} \right\|$ is bounded by (4.4), the quantity for the inverse of Φ_E is bounded by (4.5), we obtain from the equation (4.6) that

$$(4.7) \quad \left\| \frac{\partial x_2}{\partial p_1} \right\|, \left\| \frac{\partial p_2}{\partial p_1} \right\| \leq d_2 E^{-2\mu_6}.$$

It establishes certain relation between the symplectic area ω and the usual area S of the cylinder $|\omega| \geq d_3 E^{2\mu_6} |S|$.

Theorem 4.1. *We assume the conditions (H1), (H2) and (H4). There exists small $E_1 > 0$ such that for each $E \in (0, E_1]$, the cylinder $\Pi_{E, g}$ is normally hyperbolic for the map $\Phi_{\bar{G}}^{\Delta t_E}$, where $\Delta t_E = 2\lambda_1^{-1} |\ln E|$.*

Proof. The cylinder $\Pi_{E, g}$ is a 2-dimensional symplectic sub-manifold, invariant for the Hamiltonian flow $\Phi_{\bar{G}}^s$. However, it is not clear whether this cylinder is normally hyperbolic for the time-1-map $\Phi_{\bar{G}} = \Phi_{\bar{G}}^s|_{s=1}$, as it is possible that

$$m(D\Phi_{\bar{G}}|_{T\Pi}) = \inf\{|D\Phi_{\bar{G}}v| : v \in T\Pi, |v| = 1\} < 1, \quad \|D\Phi_{\bar{G}}|_{T\Pi}\| > 1,$$

and we do not have the estimate on the norm of $D\Phi_{\bar{G}}$ acting on the normal bundle.

We search for the normal hyperbolicity of $\Phi_{\bar{G}}^s$ with large s for the cylinder $\Pi_{E_0, E_1, g}$, where $0 < E_0 < E_1$, E_1 denotes the largest value such that the formulae (4.4) and (4.5) hold for each $E \leq E_1$. From these formulae, one sees that the smaller the energy reaches, the stronger hyperbolicity the map $\Phi_{E, \delta}$ obtains. It obtains such strong hyperbolicity by passing through small neighborhood of the fixed point. However, on

the other hand, the smaller the energy decreases, the longer time one needs to obtain such a return map.

Let $\Delta t_{E,k}$ denote the time interval such that, starting from $z_{E,k}^-$, the periodic orbit comes to $z_{E,k+1}^-$ after time $\Delta t_{E,k}$. In virtue of the study in Section 4, Eq. (3.10),

$$\Delta t_{E,k} \approx \tau_{E,g_\ell} - \lambda_1^{-1} \ln E, \quad \ell = i, \text{ or } i+1,$$

where both τ_{E,g_i} and $\tau_{E,g_{i+1}}$ are uniformly bounded, the choice of τ_{E,g_ℓ} depends on the path along which the periodic orbit moves. Note $\Delta t_E = \frac{2}{\lambda_1} |\ln E|$ is much larger than $\max_k \Delta t_{E,k}$. Thus, starting from any point z on the minimal periodic orbit $z_E(s)$, $\Phi_{\bar{G}}^s(z)$ passes through the neighborhood of the fixed point after time T_E . It implies that the map $\Phi_{\bar{G}}^s|_{s=T_E}$ obtains strong hyperbolicity on normal bundle.

It is used to measure the tangent hyperbolicity that the map $\Phi_{\bar{G}}$ preserves the symplectic area of the cylinder. Let us investigate how the map elongates or shortens small arc of the periodic orbit. An extremal case takes place when a short arc moves from a neighborhood of the fixed point to somewhere far away. As the orbit passes through the neighborhood of the origin $O_\delta(0)$ basically along the minimal homoclinic orbit, in a time approximately equal to $-\lambda_1^{-1} \ln \delta^{-2} E$ the variation of the length of short arc is between $O(E^{1+\mu_7})$ and $O(E^{-1-\mu_7})$. Because of the relation between the symplectic area ω and the usual area S of the cylinder, the variation of $\|D\Phi_{\bar{G}}^s\|$, restricted on the tangent bundle of the cylinder, is between $O(E^{1+\mu_7+2\mu_6})$ and $O(E^{-1-\mu_7-2\mu_6})$.

Thus, the normally hyperbolic property becomes clear: the tangent bundle of M over $\Pi_{E_0,E_1,g}$ admits $D\Phi_{\bar{G}}^s|_{s=T_E}$ -invariant splitting

$$T_z M = T_z N^+ \oplus T_z \Pi_{E_0,E_1,g} \oplus T_z N^-$$

and some $\Lambda_1 \geq 1$, $\Lambda_2 \geq 1$ and small $\nu > 0$ exist such that

$$(4.8) \quad \Lambda_1^{-1} E^{1+\nu} < \frac{\|D\Phi_{\bar{G}}^s(z)v\|}{\|v\|} < \Lambda_1 E^{-1-\nu}, \quad \forall v \in T_z \Pi_{E_0,E_1,g},$$

$$\frac{\|D\Phi_{\bar{G}}^s(z)v\|}{\|v\|} \leq \Lambda_2 E^{\frac{\lambda_2}{\lambda_1} - \nu}, \quad \forall v \in T_z N^+,$$

$$\frac{\|D\Phi_{\bar{G}}^s(z)v\|}{\|v\|} \geq \Lambda_2^{-1} E^{-\frac{\lambda_2}{\lambda_1} + \nu}, \quad \forall v \in T_z N^-,$$

hold for $s \geq \Delta t_E$ (cf. (4.4) and (4.5)). Note that $\lambda_2/\lambda_1 - \nu > 1 + \nu$ provided $\nu > 0$ is suitably small. \square

For each $E \geq E_1$, let $z_E(s)$ be the minimal periodic orbit, $\Sigma_E \subset \bar{G}^{-1}(E)$ be a 2-dimensional disk intersecting $z_E(s)$ transversally at the point $z_{E,0}$, $\Phi_E: \Sigma_E \rightarrow \Sigma_E$ be the Poincaré return map. By the generic property **(H4)**, $z_{E,0}$ is the hyperbolic fixed point of Σ_E and $\Lambda_{E_1} > 1$ exists such that

$$\|D\Phi_E(z_{E,0})v^-\| \geq \Lambda_{2,E_1} \|v^-\|, \quad \forall v^- \in T_{z_{E,0}}(W_E^- \cap \Sigma_E),$$

$$\|D\Phi_E(z_{E,0})v^+\| \leq \Lambda_{2,E_1}^{-1} \|v^+\|, \quad \forall v^+ \in T_{z_{E,0}}(W_E^+ \cap \Sigma_E).$$

As the cylinder is foliated into periodic orbits and $\Phi_{\bar{G}}$ preserves the symplectic form, some $\Lambda_{1,E_1} \geq 1$ exists such that

$$\Lambda_{1,E_1}^{-1} \|v\| \leq \|D\Phi_{\bar{G}}^s(z)v\| \leq \Lambda_{1,E_1} \|v\|, \quad \forall v^- \in T_z \Pi_{E_1,g}$$

holds for any $s > 0$. Choosing $m \in \mathbb{N}$ such that $\Lambda_{2,E_1}^m \geq 2\Lambda_{1,E_1}$, one obtains the normal hyperbolicity for $\Phi_{\bar{G}}^s(z)$ with $s \geq mT(E_1)$, where $T(E_1)$ is the period of the periodic solution in $\bar{G}^{-1}(E_1)$, $z \in \Pi_{E_1,g} \cap \bar{G}^{-1}(E)$ with $E \in [E_1, E(\nu_1, g)]$.

4.3. Persistence of cylinder: near double resonance. To apply the theorem of normally hyperbolic manifold [HPS] to the Hamiltonian G_ϵ defined in the formula (4.2), we note that the Hamiltonian $\bar{G}_\epsilon = \bar{G} + Z'_\epsilon$ is also an autonomous system with two degrees of freedom. Since $Z'_\epsilon = O(\sqrt{\epsilon})$, being aware of the non-degeneracy assumption for V (see **(H1)** and **(H2)**), we see that for each suitably small $\epsilon > 0$, the map $\Phi_{\bar{G}_\epsilon}$ admits invariant cylinder also, denoted by $\Pi_{E_0,g}$ still, with the normally hyperbolic properties (see Formulae (4.8)), independent of the size of ϵ .

Let us consider the persistence of $\Pi_{E_0,g}$ with $E_0 = \epsilon^{2d}$ with $d > 0$. Since these hyperbolic properties are posed not for the map $\Phi_{\bar{G}_\epsilon}$ but for $\Phi_{\bar{G}_\epsilon}^s$ with large s , one has to measure how large the quantity $\|\Phi_{\bar{G}_\epsilon}^s - \Phi_{G_\epsilon}^s\|$ will be.

Lemma 4.2. *Let the equation $\dot{z} = F_\epsilon(z, t)$ be a small perturbation of $\dot{z} = F_0(z, t)$, let Φ_ϵ^t and Φ_0^t denote the flow determined by these two equations respectively. Then*

$$\|\Phi_\epsilon^t - \Phi_0^t\|_{C^1} \leq \frac{B}{A}(1 - e^{-At})e^{2At}$$

where $A = \max_{t, \lambda=\epsilon, 0} \|F_\lambda(\cdot, t)\|_{C^2}$ and $B = \max_t \|(F_\epsilon - F_0)(\cdot, t)\|_{C^1}$.

Proof. Let $z_\lambda(t)$ denote the solution of the equations $\dot{z} = F_\lambda(z, t)$ for $\lambda = \epsilon, 0$ respectively, and $z_\epsilon(0) = z(0)$. Let $\Delta z(t) = z_\epsilon(t) - z(t)$, then $\Delta z(0) = 0$ and

$$\Delta \dot{z} = \partial_z F_\epsilon((\nu z + (1 - \nu)z_\epsilon)(t), t)\Delta z + (F_\epsilon - F_0)(z(t), t)$$

where $\nu = \nu(t) \in [0, 1]$. Therefore, one has

$$\|\Delta \dot{z}\| \leq \max \|\partial_z F_\epsilon\| \|\Delta z\| + \max \|F_\epsilon - F_0\|.$$

It follows from Gronwall's inequality that

$$\|\Delta z(t)\| \leq \frac{B}{A}(e^{At} - 1).$$

Along the orbit $z_\lambda(t)$, the differential of the flow Φ_λ^t obviously satisfies the equation

$$\frac{d}{dt} D\Phi_\lambda^t = \partial_z F_\lambda(z_\lambda(t), t) D\Phi_\lambda^t, \quad \lambda = \epsilon, 0.$$

Therefore, for each tangent vector v attached to $z_\lambda(0)$ one has

$$\|D\Phi_\lambda^t v\| \leq \|v\| e^{At}.$$

To study the differential of $\Phi_\epsilon^t - \Phi_0^t$, let us consider the equation of secondary variation. Let δz_λ be the solution of the variational equation $\delta \dot{z}_\lambda = \partial_z F_\lambda(z_\lambda(t), t) \delta z_\lambda$ for $\lambda = \epsilon, 0$ respectively, where $z_\lambda(t)$ solves the equation $\dot{z}_\lambda = F_\lambda(z_\lambda, t)$ and $z_\epsilon(0) = z(0)$. To measure the size $\Delta \delta z = \delta z_\epsilon - \delta z$ with the condition $z_\epsilon(0) = z(0)$, we find that

$$\begin{aligned} \left\| \frac{d(\Delta \delta z)}{dt} \right\| &\leq \max \|\partial_z F_\epsilon\| \|\Delta \delta z\| + \max \|\partial_z^2 F\| \|\Delta z(t)\| \|\delta z(t)\| \\ &\quad + \max \|\partial_z(F_\epsilon - F)\| \|\delta z(t)\| \\ &\leq A \Delta \delta z + B \|v\| e^{2At}, \end{aligned}$$

where $v = \delta z_\epsilon(0) = \delta z(0)$. By using Gronwell's inequality again, one obtains an upper bound of the variation of the differential

$$\|\Delta \delta z(t)\| \leq \frac{B}{A} \|v\| (1 - e^{-At}) e^{2At}.$$

Note that v represents initial tangent vector, it completes the proof. \square

Let us applying this lemma to the Hamiltonian G_ϵ . Treating R'_ϵ as the function of (x, p) we find that there exist some constants $a_8, a_9 > 0$ independent of ϵ such that

$$\max_s \|\Phi_{\tilde{G}_\epsilon} - \Phi_{G_\epsilon}\|_{C^1} \leq \max \left| \frac{\partial^2 R'_\epsilon}{\partial x \partial p} \right| \leq a_8 \epsilon^{(r-2)\sigma}$$

and $\max_s \|\Phi_{G_\epsilon}\|_{C^2} = \max_s \|G_\epsilon\|_{C^3} < a_9$. For $s = \frac{2}{\lambda_1} \ln \epsilon^{2d}$ one obtains from Lemma 4.2 that

$$\begin{aligned} \|\Phi_{\tilde{G}_\epsilon}^s - \Phi_{G_\epsilon}^s\|_{C^1} &\leq \frac{a_8}{a_9} \epsilon^{(r-2)\sigma} e^{\frac{4a_9}{\lambda_1} \ln \epsilon^{2d}} \\ &\leq \Lambda_3 \epsilon^{(r-2)\sigma - \frac{8a_9 d}{\lambda_1}} \end{aligned}$$

where $\Lambda_3 = a_8/a_9$. If the number d is set to satisfy the condition

$$0 < d < (r-2) \frac{\lambda_1 \sigma}{8a_9}$$

then $\|\Phi_{\tilde{G}_\epsilon}^s - \Phi_{G_\epsilon}^s\|_{C^1} \rightarrow 0$ as $\epsilon \rightarrow 0$. It allows one to apply the theorem of normally hyperbolic manifold to obtain the existence of normally hyperbolic cylinder $\Pi_{E,g}$ with $E = \epsilon^d$.

Be aware of the fact $\Pi_{E,g}$ is a cylinder with boundary, normally hyperbolic and invariant for $\Phi_{G_\epsilon}^s$, where $s = \frac{2}{\lambda_1} \ln \epsilon^{2d}$, we do not expect that the whole cylinder can survive small perturbation, it may lose some part close to the boundary. To measure to what range the cylinder remains, let us see, along each orbit of $\Phi_{G_\epsilon}^s$, how large the variation of the energy will be. Indeed, along each orbit of the flow one has

$$\left| \frac{d}{ds} G_\epsilon(z(s), s) \right| = |\partial_s G_\epsilon(z, s)| = \frac{1}{\sqrt{\epsilon}} \left| \frac{\partial R'_\epsilon}{\partial \tau} \right| \leq a_{10} \epsilon^{(r-1)\sigma - \frac{1}{2}}$$

which is small if $r \geq 5$ and $\sigma = \frac{1}{7}$. Here, the estimate $\|R'\|_{C^1} \leq O(\epsilon^{(r-1)\sigma})$ is used. Thus, if d satisfies the condition that

$$(4.9) \quad d < \min \left\{ \frac{1}{2}(r-1)\sigma - \frac{1}{4}, (r-2) \frac{\lambda_1 \sigma}{8a_9} \right\},$$

note that $\lim_{\epsilon \rightarrow 0} \epsilon^r \ln \epsilon = 0$ for any positive r , one sees that, starting from the energy level $G_\epsilon^{-1}(\epsilon^d)$, after a time of $s = \frac{2}{\lambda_1} \ln \epsilon^{2d}$, the orbit of $\Phi_{G_\epsilon}^s$ can not reach the energy level $G_\epsilon^{-1}(\epsilon^{2d})$ provided ϵ is suitably small. Indeed, one has

$$\begin{aligned} (4.10) \quad G_\epsilon(z(s), s) &\geq G_\epsilon(z(0), 0) - \int_0^s \left| \frac{d}{dt} G_\epsilon(z(t), t) \right| dt \\ &\geq \epsilon^d - \frac{2a_{10}}{\lambda_1} \epsilon^{2d} \ln \epsilon^{2d}. \end{aligned}$$

To use the theorem of normally hyperbolic invariant manifold, let us modify the perturbation near the boundary of the cylinder. Let $u: \mathbb{R} \rightarrow \mathbb{R}_+$ a smooth function

such that $u = 0$ for $t \leq 1$ and $u = 1$ for $t \geq 2$. Consider the Hamiltonian

$$G'_\epsilon = \bar{G}_\epsilon + u\left(\frac{2}{\epsilon^d}(\bar{G}_\epsilon - \epsilon^{2d}) + 1\right)R'_\epsilon$$

it coincides G_ϵ for $(x, p) \in \bar{G}_\epsilon^{-1}(E)$ with $E \geq \epsilon^d + \epsilon^{2d}$ and coincides \bar{G}_ϵ for $(x, p) \in \bar{G}_\epsilon^{-1}(E)$ with $E \leq \epsilon^{2d}$. Since $\max_s \|uR'_\epsilon(\cdot, s)\|_{C^2} \leq O(\epsilon^{(r-2)\sigma-2d})$ which is small if d satisfies the condition (4.9) and ϵ is small. Therefore, for $E = \epsilon^{2d}$, the cylinder $\Pi_{E,g}$ survives the perturbation $\Phi_{\bar{G}_\epsilon}^s \rightarrow \Phi_{G'_\epsilon}^s$ and the bottom remains invariant for $\Phi_{G'_\epsilon}^s$.

On the other hand, one has $G_\epsilon = G'_\epsilon$ for $(x, p) \in \bar{G}_\epsilon^{-1}(E)$ with $E \geq \frac{1}{2}\epsilon^d + \epsilon^{2d}$. As the function u is independent of the time s , it follows from (4.10) that the image $\Phi_{G'_\epsilon}^s(x, p) \in \bar{G}_\epsilon^{-1}(E)$ with $E > \frac{1}{2}\epsilon^d + \epsilon^{2d}$ provided $(x, p) \in \bar{G}_\epsilon^{-1}(E)$ with $E \geq \epsilon^d$, namely, one has $\Phi_{G'_\epsilon}^s(x, p) = \Phi_{G_\epsilon}^s(x, p)$. Therefore, the invariant cylinder $\Pi_{E,g}$ persists under the perturbation $\Phi_{\bar{G}_\epsilon}^s \rightarrow \Phi_{G_\epsilon}^s$ for $E = \epsilon^d$. Here, the invariance is in the sense that, emanating from each point in $\bar{G}_\epsilon \cap \Pi_{E,g}$, the orbit has to pass through the bottom of the cylinder if it eventually leaves the cylinder.

Location of Aubry set in the cylinder

Recall that the β -function is differentiable on each ray in autonomous case. For the normally hyperbolic cylinder $\Pi_{0,g}$, let $\mathbb{W} = \mathcal{L}_\beta(\cup_{\nu>0}\nu g) \subset H^1(\mathbb{T}^2, \mathbb{R})$ be the Fenchel-Legendre transformation of the ray $\cup_{\nu>0}\nu g$ with $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, it has a foliation of flats (lines). Restricted on a smooth path $\Gamma \subset \mathbb{W}$ passing through these flats transversally, the α -function is strictly convex. It follows that certain $D > 0$ exists such that

$$\alpha(c_\omega + c) - \alpha(c) \geq \langle \omega, c \rangle + D\|c\|^2$$

holds for $c_\omega, c \in \Gamma$, $\omega = \nu g$ and $c_\omega \in \mathcal{L}_\beta(\omega)$.

Under small perturbation $L \rightarrow L_\epsilon$ with $\|L' - L\|_{C^1} \leq \epsilon$, the α - and the β -function undergo small variation [Ch]

$$\begin{aligned} |\alpha_L(c) - \alpha_{L'}(c)| &\leq \epsilon, & \forall c \in H^1(M, \mathbb{R}), \\ |\beta_L(\omega) - \beta_{L'}(\omega)| &\leq \epsilon, & \forall \omega \in H_1(M, \mathbb{R}). \end{aligned}$$

Consequently, let α_{G_ϵ} denote the α -function for the Hamiltonian G_ϵ , then

$$(4.11) \quad \alpha_{G_\epsilon}(c_\omega + c) - \alpha_{G_\epsilon}(c) \geq \langle \omega, c \rangle + D\|c\|^2 - 2\epsilon^{(r-1)\sigma}$$

holds for the cohomological classes on the path.

Let $\Pi_{E,g,\epsilon}$ be the invariant cylinder invariant for the symplectic diffeomorphism $\Phi_{G'_\epsilon}^s$ with $s = 0 \bmod \sqrt{\epsilon}$ (the Hamiltonian G_ϵ is time- $\sqrt{\epsilon}$ periodic). For any $c \in \Gamma$, the Aubry set stays entirely in the cylinder in the following sense: for each c -static curve γ , the orbit in the phase space $(x(s), p(s) = \partial_x L_{G'_\epsilon}(\gamma(s), \dot{\gamma}(s), s) \in \Phi_{G'_\epsilon}^s$ for $s = 0 \bmod \sqrt{\epsilon}$. In this sense, one has

Lemma 4.3. *For each $c \in \Gamma \cap \alpha^{-1}(E)$ with $E \geq \epsilon^{d/2}$, any orbit in the Aubry set does not hit the energy level set $G_\epsilon^{-1}(E)$ with $E \leq \epsilon^d$.*

Proof. Let us assume the contrary: there exists an orbit $(x(s), p(s))$ in this Aubry set which hits the energy level $G_\epsilon^{-1}(\epsilon^d)$ at the time $s = 0$. As the orbit stays entirely in the invariant cylinder and the perturbation is of order $\epsilon^{(r-1)\sigma}$, it come back to the

neighborhood of $(x(0), y(0))$ in a time $S = \lambda_1^{-1} \ln \epsilon^d + \tau_\epsilon$ where $s = 0 \bmod \sqrt{\epsilon}$ and τ_ϵ remains bounded as $\epsilon \rightarrow 0$:

$$\|(x(s), y(s)) - (x(0), y(0))\| \leq d_0(\epsilon^{2d} \ln \epsilon^d + \sqrt{\epsilon})\epsilon^{-d\nu}.$$

One obtains it by using the estimate

$$|G'_\epsilon(x(s), p(s), s) - G'_\epsilon(x(0), p(0), 0)| \leq \int_0^s \left| \frac{d}{dt} G_\epsilon(z(t), t) \right| dt \leq d_1 \epsilon^{2d} \ln \epsilon^d.$$

and being aware of the cylinder is crumpled up to the order $O(E^{-\nu})$ (see (4.7)). As $\epsilon^\nu \ln \epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, by assuming $d \leq 1/4$ we let $d' = (1 - \nu)d$. Since the curve $x(s)$ is c -static we have

$$(4.12) \quad \left| \int_0^S (L_{G_\epsilon}(x(t), \dot{x}(t), t) - \langle c, \dot{x}(s) \rangle + \alpha_{G_\epsilon}(c)) dt \right| \leq d_2 \epsilon^{2d'}.$$

Consider an orbit $(x'(s), p'(s))$ in the Aubry set for $c' \in \Gamma \cap \alpha^{-1}(\epsilon^d)$. This orbit must hit the energy level set $G_\epsilon^{-1}(\epsilon^d)$ at the time $|s_0| \leq \sqrt{\epsilon}$. This orbit also comes back to a small neighborhood of this point in a time S' approximately equal to $\lambda_1^{-1} \ln \epsilon^d + \tau_\epsilon$. Indeed, starting from the point $(x'(s_0), p'(s_0))$ the orbit comes back to its small neighborhood in a time of order $O(|\ln \epsilon^d|)$, along this piece of the orbit the variation of the energy is at most of order $O(\epsilon^{2d} |\ln \epsilon^d|)$. As the cylinder crumples at most of order $O(\epsilon^{-\nu})$, the speed along these two orbit remains almost the same, namely,

$$\|\dot{x}(s) - \dot{x}'(s')\| \leq \epsilon^{2d-\nu} |\ln \epsilon^d| \quad \text{provided} \quad x_1(s) = x'_1(s').$$

It produces $|S - S'| \leq O(\epsilon^{2d} |\ln \epsilon^d|^2)$. As $x'(s)$ is c' -static, we have

$$(4.13) \quad \left| \int_0^{S'} (L_{G_\epsilon}(x'(t), \dot{x}'(t), t) - \langle c', \dot{x}'(s') \rangle + \alpha_{G_\epsilon}(c')) dt \right| \leq d_2 \epsilon^{2d'}.$$

One can easily see that

$$\left| \int_0^{S'} (L_{G_\epsilon}(x'(t), \dot{x}'(t), t) dt - \int_0^S (L_{G_\epsilon}(x(t), \dot{x}(t), t) dt) \right| \leq d_3 \epsilon^{2d-\nu} |\ln \epsilon^d|^2.$$

and

$$\begin{aligned} \left\| \frac{\bar{x}(S) - \bar{x}(0)}{S} - \omega' \right\| &\leq d_4 \epsilon^{2d-\nu} |\ln \epsilon^d|, \\ \left\| \frac{\bar{x}'(S + s_0) - \bar{x}'(s_0)}{S'} - \omega' \right\| &\leq d_4 \epsilon^{2d-\nu} |\ln \epsilon^d|, \end{aligned}$$

where $\omega' \in \mathcal{L}_\beta^{-1}(c')$, \bar{x} and \bar{x}' denote the lift of the curve x and x' respectively. Subtracting (4.13) from (4.12) one then obtains that

$$(4.14) \quad |\alpha_{G_\epsilon}(c) - \alpha_{G_\epsilon}(c') - \langle c - c', \omega' \rangle| \leq d_5 \epsilon^{2d-\nu} |\ln \epsilon^d|.$$

On the other hand, as $\|c - c'\| \geq d_6 |\alpha_{G_\epsilon}(c) - \alpha_{G_\epsilon}(c')| = \frac{1}{2} \epsilon^{d/2}$ holds for $c, c' \in \Gamma$, one obtains from (4.11) that

$$\alpha_{G_\epsilon}(c) - \alpha_{G_\epsilon}(c') - \langle c - c', \omega \rangle \geq d_6 \epsilon^d.$$

It contradicts (4.14). This completes the proof. \square

For $d > 0$ satisfying the condition (4.9), going back to the original coordinates $(E \rightarrow \epsilon E, y = \sqrt{\epsilon} p \text{ and } s = \sqrt{\epsilon} \tau)$, we obtain

Theorem 4.2. *Given a homology class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, there exist a wedge-shaped channel $\mathbb{W} \subset H^1(\mathbb{T}^2, \mathbb{R})$ and a cylinders $\Pi \subset \mathbb{R}^2 \times \mathbb{T}^2$ such that for $\epsilon \leq E_0^{1/d}$*

1, this wedge-shaped channel reaches to certain small neighborhood of the flat \mathbb{F}_0 in the sense

$$\min_{c \in \mathbb{W}} \alpha(c) - \min_{c \in H^1(\mathbb{T}^2, \mathbb{R})} \alpha(c) = D_\alpha \epsilon^{1+d},$$

where $D_\alpha > 0$ is independent of ϵ . For each $c \in \mathbb{W}$ the Mather set is contained in the cylinder: $\tilde{\mathcal{M}}_0(c) \subset \Pi$. For each $c \in \mathbb{W}$ with $\alpha(c) \geq D_\alpha \epsilon^{1+d/2}$, the Aubry set stays in the cylinder.

2, the cylinder Π is invariant and normally hyperbolic for the Hamiltonian flow Φ_G^t , $H_1(\Pi, \mathbb{Z}) = g\mathbb{Z}$. The invariance is in the sense that any orbit must pass through the bottom of the cylinder $\Sigma \cap G^{-1}(D_\alpha \epsilon^{1+d})$ if, starting from the interior, it leaves the cylinder.

Let us consider the autonomous Hamiltonian H with the form of (4.1). Let $E_0 > \min \alpha_H$, $G(x, y, \tau)$ be the function solves the equation $H(x, y, x_3, G(x, y, -x_3)) = E_0$. Then G has the form of (4.1). Applying Theorem 3.4 one obtains

Theorem 4.3. *Assume $E_0 > \min \alpha_H$. For each $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, there exists an 3-dimensional cylinder $\bar{\Pi}_{\epsilon, g} \subset H^{-1}(E_0)$ as well as a wedge-shaped region $\mathbb{W}' \subset \alpha^{-1}(E_0)$ such that*

1, $\bar{\Pi}_{\epsilon, g}$ is a small deformation of $\bar{\Pi}_{\epsilon, g}^0$ where

$$\bar{\Pi}_{\epsilon, g}^0 = \{(x_\nu(x_3 + x_3^*), y_\nu(x_3 + x_3^*), x_3, G_T(x_\nu, y_\nu)) : [x_\theta] = g, x_3, x_3^* \in \mathbb{T}\},$$

G_T solves the equation $H_T(x, y, G_T(x, y)) = E_0$. The small deformation is in the sense that

$$d_H(\bar{\Pi}_{\epsilon, g}^0, \bar{\Pi}_{\epsilon, g}) \leq D_\alpha \epsilon^{1+d}$$

where d_H denotes the Hausdorff distance between two sets;

2, $\bar{\Pi}_{\epsilon, g}$ is invariant for the Lagrange flow ϕ_L^t , namely, for each $(x, \dot{x}) \in \bar{\Pi}_{\epsilon, g}$, $\phi_L^t(x, \dot{x}) \notin \bar{\Pi}_{\epsilon, g}$ provided $\exists t_0 \leq t$ such that $\phi_L^t(x, \dot{x})$ is at the boundary of $\bar{\Pi}_{\epsilon, g}$;

3, for each $c \in \text{int} \mathcal{L}_{\beta_L}(\Gamma_{\omega, i})$ with $c_3 \geq D_\alpha \epsilon^{1+d/2}$, the Aubry set is contained in that cylinder $\tilde{\mathcal{N}}(c) \subset \bar{\Pi}_{\epsilon, g}$;

4, $\bar{\Pi}_{\epsilon, g}$ is normally hyperbolic for ϕ_L^t with $t = \frac{2d \ln \epsilon}{\lambda_1 \sqrt{\epsilon}}$.

Finally, let us consider all rotation vectors νg with $\nu \in (0, \nu^*]$. It is generic that there are finitely many bifurcation points, denoted by $\nu_1, \nu_2, \dots, \nu_m$. For each point $0 < \nu \neq \nu_i$, the c -minimal measure is uniquely supported on a periodic orbit $d\gamma_\nu$, where $c \in \mathcal{L}_{\beta_G}(\nu g)$. At those bifurcation points ν_i , the Mather set is composed by two minimal periodic orbits $d\gamma_{\nu_i}^-$ and $d\gamma_{\nu_i}^+$. Let z_ν be the periodic orbit determined by γ_ν in the phase space. For $\nu \in (\nu_i, \nu_{i+1})$, these periodic orbits constitute a piece of cylinder Π_i bounded by $z_{\nu_i}^+$ and $z_{\nu_{i+1}}^-$. As all these periodic orbits are assumed hyperbolic, the cylinder Π_i can be extended a little bit to contain periodic orbits which are hyperbolic also, but do not support minimal measure. They are local minimal.

These cylinders are obviously invariant and normally hyperbolic for the Hamiltonian flow $\Phi_{\bar{G}}^t$ with large t . By applying the theorem of normally hyperbolic manifold, one can see that there exists some $\Pi_{i,\epsilon}$ which is invariant for $\Phi_{\bar{G}}^k$ for large $k \in \mathbb{Z}$ and keeps very close to Π_i . Restricted on each of these cylinders, $\Phi_{\bar{G}}$ is area-preserving and twist.

Consider a path $\Gamma_g: [0, 1] \rightarrow \mathbb{C}_g = \cup_{\nu \in [\nu_1 - \delta, \nu^*]} \mathcal{L}_\beta(\nu g)$. Except for finitely many $\nu_{i,\epsilon}$ very close to ν_i , the time-1-section of the Mather set for rotation vector νg corresponds to an Aubry-Mather set in one of these cylinders, and for the rotation $\nu_i g$ it consists of two.

4.4. Transition from double to single resonance. Some normally hyperbolic invariant cylinder has been shown to reach $O(\sqrt{\epsilon}^{1+\frac{d}{2}})$ -neighborhood of the double resonant point. This cylinder extends to the place a bit far away from the double resonant point. To see how to transit from double resonance to single resonance, let us homogenize the Hamiltonian in a region $\|y - y_0\| \leq O(\sqrt{\epsilon})$ and changes y_0 . Such approach relies on the following:

Proposition 4.2. *For nearly integrable Lagrangian $L(x, \dot{x}, t) = \ell(\dot{x}) + \epsilon \ell_1(x, \dot{x}, t)$, each orbit in Mather set $(\gamma, \dot{\gamma})$*

$$\|\dot{\gamma}(t) - \dot{\gamma}(0)\| \leq O(\sqrt{\epsilon}), \quad \forall t \in \mathbb{R}.$$

The result is proved in [BK] for time-1-map. It is then obviously true for the Hamiltonian flow. It follows that $\|y(t) - y(0)\| \leq O(\sqrt{\epsilon})$. It makes possible for us to homogenize the Hamiltonian in the range $\|y - y_0\| \leq d\sqrt{\epsilon}$ with suitably large $d > 0$. Also using new variable $y - y_0 = \sqrt{\epsilon}p$ and $s = \sqrt{\epsilon}\tau$, the homogenized Hamiltonian equation turns out to be the following form

$$\frac{dx}{ds} = \frac{\omega}{\sqrt{\epsilon}} + Ap, \quad \frac{dp}{ds} = -\frac{\partial V}{\partial x}(x, y_0),$$

where $A = \partial^2 h(y_0)$, $\omega = \partial h(y_0)$ satisfies a resonant condition. This ω determines an integer K such that $K\omega$ is an irreducible integer vector g_ω . The corresponding Lagrangian reads

$$L(\dot{x}, x) = \frac{1}{2} \left\langle A^{-1} \left(\dot{x} - \frac{\omega}{\sqrt{\epsilon}} \right), \left(\dot{x} - \frac{\omega}{\sqrt{\epsilon}} \right) \right\rangle - V(x)$$

With the potential $V(x)$ on the torus one associates its time average $[V]$ along the orbits of the linear flow defined by $\omega: x \rightarrow x + \omega t$

$$[V](x) = \frac{1}{T} \int_0^T V(x + \omega t) dt,$$

where T is the period of the frequency ω . The function $[V]$ is then defined on a circle. Let x_0 be the minimal point of $[V]$, it is a circle $\Gamma \subset \mathbb{T}^2$. The averaged Hamiltonian is also associated with a Lagrangian

$$[L](\dot{x}, x) = \frac{1}{2} \left\langle A^{-1} \left(\dot{x} - \frac{\omega}{\sqrt{\epsilon}} \right), \left(\dot{x} - \frac{\omega}{\sqrt{\epsilon}} \right) \right\rangle - [V](x).$$

Let $T_{\omega,\epsilon} = K\sqrt{\epsilon}$ be the period of the frequency $\omega/\sqrt{\epsilon}$, $[\gamma]_{\omega,\epsilon}: [0, T_{\omega,\epsilon}] \rightarrow \mathbb{T}^2$ be the minimizer of the action

$$\inf_{[\xi]=g_\omega} \int_0^{T_{\omega,\epsilon}} [L](d\xi(s)) ds,$$

then it is a curve of maximal points of $[V]$ with constant speed $\dot{\gamma}_{\omega,\epsilon} = \omega/\sqrt{\epsilon}$. Consider $[V]$ as a function defined on \mathbb{T}^2/Γ and denote by $[V]''$ the second derivative for the variable of \mathbb{T}^2/Γ .

Recall the normal form we obtained by several steps of KAM iteration. It remains valid in the region $\|y - y_0\| = O(\epsilon^\sigma/K)$, where y_0 is a double resonant point, $\sigma \in (\frac{1}{3}, \frac{1}{7})$. Thus, the valid domain for the new action variable is $\|p\| \leq O(\epsilon^{-\lambda})$ with $\lambda \in (\frac{1}{6}, \frac{5}{14})$. Let $\gamma_{\omega,\epsilon}: [0, T_{\omega,\epsilon}] \rightarrow \mathbb{T}^2$ be the minimizer of the action

$$\inf_{[\xi]=g_\omega} \int_0^{T_{\omega,\epsilon}} L(d\xi(s))ds,$$

and let $\Gamma + \delta$ denotes a translation of Γ such that $d(\Gamma + \delta, \Gamma) = \delta$, then we have

Proposition 4.3. *Assume $-[V]$ is non-degenerate at its minimal point $-[V]'' > \Lambda$, and assume some $\lambda > 0$ exists so that $T_{\omega,\epsilon} = \epsilon^\lambda$. Then some constant $D, D' > 0$ exists such that the minimizer $\gamma_{\omega,\epsilon}$ entirely stays in $D\epsilon^\lambda$ -neighborhood of the circle $\Gamma + \delta$, i.e. $d(\gamma(s), \Gamma + \delta) < D\epsilon^\lambda$ holds for each $s \in [0, \epsilon^\lambda]$ and $|\delta| \leq D'\epsilon^{\lambda/2}$.*

Proof. Since the minimizer solves the Lagrange equation, its second derivative remains bounded $\|\ddot{\gamma}_{\omega,\epsilon}\| < a_1$. Thus, as the first step, we claim that the minimizer stays entirely in $D\epsilon^\lambda$ -neighborhood of $\Gamma + d$, a translation of the circle Γ . Otherwise, there would be a point on the minimizer where $\|\dot{\gamma}_{\omega,\epsilon} - \omega/\sqrt{\epsilon}\| \geq a_2 D$. Since the potential is bounded $|V| < a_3$, one obtains that $A_L(\gamma_{\omega,\epsilon}) \geq (a_4 D - a_{13})\epsilon^\lambda$. On the other hand, the action along the curve $\gamma(t) = x_0 + \omega/\sqrt{\epsilon}t$ would be not bigger than $a_3\epsilon^\lambda$. Thus, $\gamma_{\omega,\epsilon}$ would be not a minimizer if $D > 2a_4^{-1}a_3$.

Let us compare the action of L along the curve $\gamma_{\omega,\epsilon}$ with that along the curve $[\gamma]_{\omega,\epsilon}$. Since $d(\gamma_{\omega,\epsilon}(s), \Gamma) \geq a_5\epsilon^{\lambda/2}$, some $x_1 \in \mathbb{T}^2/\Gamma$ exists such that

$$|\gamma_{\omega,\epsilon}(t) - (x_1 + \omega'\sqrt{\omega}t)| \leq a_6\epsilon^\lambda, \quad |(x_1 - x_0)/\Gamma| \geq D'\epsilon^{\lambda/2}.$$

It follows that

$$\begin{aligned} A(\gamma_{\omega,\epsilon}) - A([\gamma]_{\omega,\epsilon}) &= \frac{1}{2} \int_0^{\epsilon^\lambda} \left\langle A^{-1}\left(\dot{\gamma}_{\omega,\epsilon}(t) - \frac{\omega}{\sqrt{\epsilon}}\right), \left(\dot{\gamma}_{\omega,\epsilon}(t) - \frac{\omega}{\sqrt{\epsilon}}\right) \right\rangle dt \\ &\quad - \int_0^{\epsilon^\lambda} \left(V(\gamma_{\omega,\epsilon}(t)) - V(x_0 + \omega/\sqrt{\epsilon}t) \right) dt \\ &> - \int_0^{\epsilon^\lambda} \left(V(\gamma_{\omega,\epsilon}(t)) - V(x_1 + \omega/\sqrt{\epsilon}t) \right) dt \\ &\quad + (-[V](x_1) + [V](x_0))\epsilon^\lambda \\ &> \left(\frac{1}{2}\Lambda D'^2 - a_6 |\max \partial V| \right) \epsilon^{2\lambda} \end{aligned}$$

which is positive provided $D' > 0$ is suitably large, but it contradicts the minimality of the curve $\gamma_{\omega,\epsilon}$. \square

Note the picture of minimal periodic orbit close to double resonance, one can see from this proposition how the shape of the periodic orbit changes when it moves away from double resonance to single resonance.

5. ANNULUS OF INCOMPLETE INTERSECTION

Let us also start with the Hamiltonian G defined by Formula (4.2), it has two and half degrees of freedom. Given any two homology class $g, g' \in H_1(\mathbb{T}^2, \mathbb{Z})$, The theorem 4.2 confirms the existence of two wedge-shaped regions \mathbb{W} and \mathbb{W}' which reach to the boundary of the annulus

$$\mathbb{A}_0 = \left\{ c \in H^1(M, \mathbb{R}) : 0 < \alpha(c) - \min \alpha < D\epsilon^{1+d'} \right\},$$

For each class in \mathbb{W} and \mathbb{W}' , the Aubry set lies in the normally hyperbolic cylinder and can be connected to other Aubry set lying in the cylinder also. However, it seems unclear whether these two wedges can reach to the flat \mathbb{F}_0 . Thus, a notable difficulty rises as these cylinders are separated by \mathbb{A}_0 around the flat \mathbb{F}_0 , it is the problem of crossing double resonance.

It is the goal of this section to find an annulus $\mathbb{A} \supsetneq \mathbb{A}_0$ where those two wedge-shaped regions are plugged into and for each class in that annulus, the stable set of the Aubry set “intersects” the unstable set non-trivially, possibly incomplete. In other words, for each class in this region, the Mañé set does not cover the whole configuration space.

5.1. The Mañé set for $c \in \partial^* \mathbb{F}_0$. As the first step, let us consider the homogenized Hamiltonian \bar{G} and investigate all cases when the Mañé set covers the whole configuration space.

For each $c \in \partial^* \mathbb{F}_0$, there are at least two ergodic minimal measures μ, μ_c and some c -static curve ξ_c exists such that

$$\text{supp} \mu_c = \overline{\bigcup_{t \in \mathbb{R}} (\xi_c(t), \dot{\xi}_c(t))}.$$

Remember that μ is supported on the fixed point $(x, y) = 0$. If there are two different ergodic measures μ_c and μ'_c , the origin $x = 0$ lies in a strip bounded by c -static curves ξ_c and ξ'_c . Some $d > 0$ exists such that B_d is contained in the interior of the strip. By suitably choosing finite covering space one can assume the existence of these two curves. Let U_c^\pm and U'_c^\pm denote the elementary weak KAM solution determined by ξ_c and ξ'_c respectively, we investigate what happens when $U_c^- - U'_c^+ = 0$ holds in this strip. As the configuration space is two dimensional, for each x in this region, $(x, y) = (x, \partial U_c^-(x)) = (x, \partial U'_c^+(x))$ uniquely determines a c -semi static curve which lies entirely in this strip. The c -semi static curves considered in this subsection are determined by $U_c^- = U'_c^+$. It is possible that some curve approaches to the origin as $t \rightarrow \infty(-\infty)$, in this case, it approaches to the curve ξ_c (ξ'_c) as $t \rightarrow -\infty(\infty)$.

Supported on the fixed point, the minimal measure μ is minimal for all $c \in \mathbb{F}_0$. Thus, there always exists some semi-static curve γ_c^\pm connecting the fixed point to the support of μ_c for each $c \in \partial^* \mathbb{F}_0$

$$\lim_{t \rightarrow \pm\infty} \gamma_c^\pm(t) = 0, \quad \text{and} \quad \lim_{t \rightarrow \mp\infty} \gamma_c^\pm(t) \rightarrow \text{supp} \mu_c.$$

As all eigenvalues are assumed different, it is generic that all minimal homoclinic curves approach to the fixed point in the direction $\Lambda_{1,x}$, associated to the smallest

eigenvalue, in the sense that

$$\lim_{t \rightarrow \pm\infty} \frac{\dot{\gamma}_{c_i}^\pm(t)}{\|\dot{\gamma}_{c_i}^\pm(t)\|} = \pm\Lambda_{1,x}.$$

But this does not exclude the possibility that some c -semi static curves approach to the point in the direction of $\Lambda_{2,x}$. It provides us a criterion to classify the cases when the Mañé set covers the configuration manifold.

Case 1: no c -static curve approaches the origin in the direction of $\Lambda_{1,x}$. As $|\lambda_1| < |\lambda_2|$, in this case, there exist exactly two static curves γ_c^\pm such that $\gamma_c^\pm(t) \rightarrow 0$ as $t \rightarrow \pm\infty$. They approach the origin in the direction $\Lambda_{2,x}$ and

$$\lim_{t \rightarrow \infty} \frac{\dot{\gamma}_c^+(t)}{\|\dot{\gamma}_c^+(t)\|} = \lim_{t \rightarrow -\infty} \frac{\dot{\gamma}_c^-(t)}{\|\dot{\gamma}_c^-(t)\|}.$$

Other cases are classified under the condition that there exist some c -static curves approaching to the origin in the direction of $\Lambda_{1,x}$. Since the curves ξ_c as well as ξ'_c is disjoint with the origin, some number $\delta > 0$ exists such that these two curves do not hit the ball $B_\delta(0)$. Of course, δ may depend on c . Let γ_c^+ be such a curve approaching the origin as $t \rightarrow \infty$, it intersects the circle $\partial B_\delta(0)$ at some point. Let $I^\pm \subset \partial B_\delta(0)$ be such a set that passing through each point $x \in I^+$ a unique c -static curve $\gamma_{c,x}$ approaches to the origin, as $t \rightarrow \pm\infty$, in the direction of $\Lambda_{1,x}$. By assumption, the set I^+ is not empty. Obviously, I^+ does not occupy the whole circle and can be made closed by adding at most two points, through which some semi-static curves approach the origin in the direction of $\Lambda_{2,x}$.

For a point $x \in \partial B_\delta(0)$ very close to I^+ , there is a unique c -semi static curve, determined by $U_c^- = U_c'^+$, passing through this point. Because of Proposition 3.2, the curve γ_c gets very close to the origin and leaves in a direction far away from $\Lambda_{1,x}$. Let I_i^+ be a connected component of I^+ , it may be a point or an interval. If it is a point, the c -semi static curve approaches the origin in the direction of $\Lambda_{2,x}$. Let $x_i, x'_i \in \partial B_\delta(0)$ be two sequence of points such that x_i approaches I_i^+ from one side and x'_i approaches I_i^+ from other side. Let γ_i and γ'_i be the c -semi static curves determined by x_i and x'_i respectively, then some $x^-, x'^- \in \partial B_\delta(0)$ exist such that γ_i (γ'_i) approaches to x^- (x'^-) respectively as $i \rightarrow \infty$.

If $x^- = x'^-$, it corresponds to a c -semi static curve which approaches to the origin as $t \rightarrow -\infty$. Clearly, it approaches in the direction of $\Lambda_{2,x}$, guaranteed by Proposition 3.2. This leads to

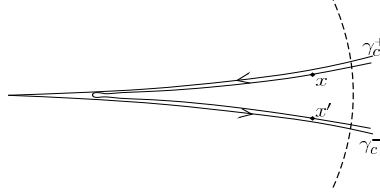
Case 2: there exists exactly one c -semi static curve approaching to origin in the direction of $\Lambda_{2,x}$.

If $x^- \neq x'^-$, let I_i^- denote the arc bounded by these two points and does not contain I_i^+ . One can see from the proof of Proposition 3.2 that the angle of this arc is not smaller than $\pi/2$. Passing from each point in the interior of the arc, the c -semi static curve approaches to the origin as $t \rightarrow -\infty$. Obviously, these curves constitute a sector where $U_c^- = U_c'^+ = U^-$, as the orbits determined by these curves entirely lie in the unstable manifold of the fixed point. Since the fixed point is hyperbolic, it has its stable and unstable manifolds W_0^\pm and some $d > 0$ exists such that W_0^\pm keeps horizontal in $B_d(0)$, namely, some generating function U^\pm exists such

that $W_0^\pm|_{B_d(0)} = \text{graph} dU^\pm|_{B_d(0)}$. Therefore, the size of the sector-shaped region is independent of the size of δ . This leads to

Case 3: in the disk $B_d(0)$ there is a sector-shaped region with the filed angle not smaller than $\pi/2$, where $U_c^- = U_c^+ = U^-$.

Let γ_c^+ (γ_c^-) be c -semi static curve passing through a point in I_i^+ (I_i^-) respectively, then they approach to the origin in opposite direction as $t \rightarrow \pm\infty$ respectively, i.e. $\lim_{t \rightarrow \infty} \dot{\gamma}_c^+(t) \|\dot{\gamma}_c^+(t)\|^{-1} = \lim_{t \rightarrow -\infty} \dot{\gamma}_c^-(t) \|\dot{\gamma}_c^-(t)\|^{-1}$. To verify this claim, let us assume the contrary. Thus, these two curves cut the ball B_δ into two parts, one is a sharp wedge-shaped, denoted by W . We choose a c -semi static curve lying in W and keeping very close to the curves γ_c^- and γ_c^+ . In canonical coordinates such that $\bar{G} = \frac{1}{2}(p_1^2 - \lambda_1 x_1^2) + \frac{1}{2}(p_2^2 - \lambda_2 x_2^2) + O(\|(x, p)\|^3)$, the set W has a vertex at the origin and for each point $x = (x_1, x_2) \in W$, one has $|x_2| \leq \theta|x_1|$ where $\theta > 0$ is very small. Since the fixed point is hyperbolic, it has local stable and unstable manifold, deter-



mined by the generating functions U^+ and U^- respectively. Restricted in W , these functions satisfy the condition

$$U^-(x) - U^-(0) \geq \frac{\lambda_1^2}{3} \|x\|^2, \quad U^+(0) - U^+(x) \geq \frac{\lambda_1^2}{3} \|x\|^2, \quad \forall \|x\| \leq \delta.$$

Pick up two points x and x' very close to γ_c^\pm respectively, through which some c -semi static curve γ_c passes, namely, some $t' > t$ exist such that $\gamma_c(t) = x$ and $\gamma_c(t') = x'$. Note the orbit determined by γ_c^+ (γ_c^-) lies in the stable (unstable) manifold, by definition ones has

$$A[\gamma_c|_{[t, t']}] \geq \frac{3}{4} U_c^-(x') - U_c^+(x) \geq \frac{\lambda_1^2}{4} (\|x'\|^2 + \|x\|^2).$$

If we choose x sharing the same first coordinate with x' and connect them with a straight line $\zeta: [0, \theta] \rightarrow \mathbb{T}^2$, then $|\dot{\zeta}| \leq O(1)$ and the action along this curve one has $A[\zeta] \leq O(\theta)$. It contradicts the minimality of γ_c , thus the claim is proved.

It follows that I^+ has only one connected component in this case. If not, let I_k^+ be another connected component adjacent to I_i^+ . Let γ_k be a c -semi static curve passing through I_k^+ and let γ_i be also a c -semi static curve passing through a point between I_i^+ and I_k^+ and very close to I_i^+ . By definition, γ_k approaches to origin as $t \rightarrow \infty$, it implies that γ_k would intersect γ_i somewhere near the origin. It is absurd. Thus, we obtain the left picture in Figure 6.

By similar argument applying to the set I^- , we have either the case 2 again or

Case 4: in the disk $B_d(0)$ there is a sector-shaped region with the filed angle not smaller than $\pi/2$, where $U_c^- = U_c^+ = U^+$, see the right picture in Figure 6.

We claim that all of these cases do not occur for generic potential V . The first two cases takes place at most for four invariant measures, as there are four curves

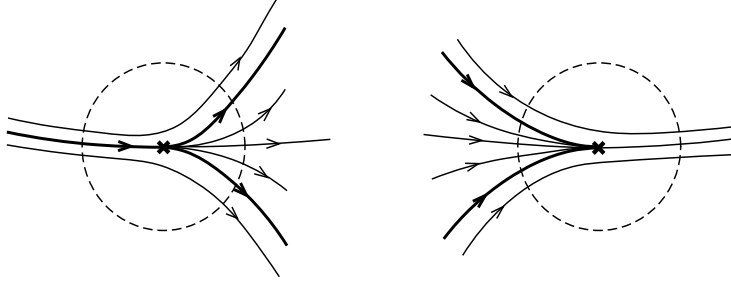


FIGURE 6.

only which approaches to the origin in the direction of $\Lambda_{2,x}$. Each of these curves approaches at most to one Mather set. These Mather set corresponds to at most four sub-flats. Let $V_\delta - V$ be a non-negative function such that its support does not touch these four curves as well as the support of the minimal measure. By perturbing the potential $V \rightarrow V_\delta$, one can see that Mañé set does not cover \mathbb{T}^2 for these cohomology classes.

The case 3 and 4 take place for at most four Mather sets also, as each sector-shaped region has the field angle bigger than $\pi/2$, and the orbit determined by $(x, y) = (x, \partial_x U^\pm)$ can approaches to one Mather set only. Let us destruct it one by one. If some sector-shaped region $S^+ \subset B_d(0)$ exists where $U_c^- = U_c'^+ = U^+$, consequently, $\mathcal{M}(c) \cap S^+ = \emptyset$. We divide it into three sub-sectors $S^+ = S_1^+ \cup S_2^+ \cup S_3^+$, each of which is composed by c -semi static curves approaching to the origin as $t \rightarrow \infty$ and S_1^+ is disjoint with S_3^+ . We introduce another potential V_δ such that the function $V_\delta - V$ is non-negative, $\text{supp}(V_\delta - V) \subset S_2^+ \setminus B_{d_1}$ ($d_1 < d$).

For the perturbed Hamiltonian $\bar{G}_\delta = \frac{1}{2}\langle Ap, p \rangle + V_\delta(x)$, the minimal measure for the class c is the same as that for unperturbed Hamiltonian. Let $U_{c,\delta}^-$, $U_{c,\delta}'^+$ be the elementary weak KAM solutions of the perturbed Hamiltonian, associated to the minimal measure μ_c and μ_c' respectively, one has

$$\arg \min(U_{c,\delta}^- - U_{c,\delta}'^+) \cap \text{supp}(V_\delta - V) = \emptyset, \quad \arg \min(U_{c,\delta}^- - U_{c,\delta}'^+) \supset S_1^+ \cup S_3^+.$$

Under such perturbation, there might be another cohomology class c' such that $U_{c'}^- - U_{c'}'^+ = 0$ holds on the whole torus and a sector S^- exists where $U_{c'}^- = U_{c'}'^+ = U^-$. Clearly, $\mathcal{M}(c') \cap S^- = \emptyset$. Again, we divide it into three sub-sectors $S^- = S_1^- \cup S_2^- \cup S_3^-$, each of which is composed by c' -semi static curves approaching to the origin as $t \rightarrow -\infty$ and S_1^- is disjoint with S_3^- . We introduce again a perturbed potential V_δ such that the function $V_\delta - V$ is non-negative, $\text{supp}(V_\delta - V) \subset S_2^- \setminus B_{d_1}$ ($d_1 < d$).

For the new perturbed Hamiltonian, the minimal measure for the class c' is the same as that for unperturbed Hamiltonian. Let $U_{c',\delta}^-$, $U_{c',\delta}'^+$ be the elementary weak KAM solutions of the perturbed Hamiltonian, associated to the minimal measure $\mu_{c'}$ and $\mu_{c'}'$ respectively, one has

$$\arg \min(U_{c',\delta}^- - U_{c',\delta}'^+) \cap \text{supp}(V_\delta - V) = \emptyset, \quad \arg \min(U_{c',\delta}^- - U_{c',\delta}'^+) \supset S_1^- \cup S_3^-.$$

For suitably small $d > 0$, the Hamiltonian flow determined by \bar{G} is well approximated by its linearized flow when they are restricted in the ball $B_d(0)$. While for the linearized flow, one obtains an orbit in the stable manifold $(x(-t), -y(-t))$ if

$(x(t), y(t))$ is an orbit in the unstable manifold. Therefore, by choosing S_2^- carefully, one can find $\check{S}_k^\pm \subset S_k^\pm$ ($k = 1, 3$) such that each \check{S}_k^+ is composed by c -semi static curves which approach to the origin as $t \rightarrow \infty$ with

$$\arg \min(U_{c,\delta}^- - U_{c,\delta}^+) \supset \check{S}_1^+ \cup \check{S}_3^+, \quad \arg \min(U_{c,\delta}^- - U_{c,\delta}^+) \cap \text{supp}(V_\delta - V) = \emptyset,$$

each \check{S}_k^- is composed by c' -semi static curves which approach to the origin as $t \rightarrow -\infty$ and

$$\arg \min(U_{c',\delta}^- - U_{c',\delta}^+) \supset \check{S}_1^- \cup \check{S}_3^-, \quad \arg \min(U_{c',\delta}^- - U_{c',\delta}^+) \cap \text{supp}(V_\delta - V) = \emptyset.$$

Since there are at most two sectors corresponding to unstable manifold and two sectors corresponding to stable manifold, there are at most four pairs of minimal measure (μ_{c_i}, μ'_{c_i}) ($i = 1, 2, 3, 4$) for which the case 3 and 4 takes place. By the construction of potential in this way, one can claim that

Lemma 5.1. *It is an open and dense condition for the potential V that for all class $c \in \partial^* \mathbb{F}_0$, the Mañé set does not cover the torus: $\mathcal{N}(c) \subsetneq \mathbb{T}^2$.*

Indeed, these four pairs of minimal measure (μ_{c_i}, μ'_{c_i}) ($i = 1, 2, 3, 4$) corresponds to four sub-flats contained in $\partial^* \mathbb{F}_0$ (each sub-flat may be just a point). For any other class $c \in \partial^* \mathbb{F}_0$, the Mañé set can not cover the whole torus also. Otherwise, there would be a sector S^\pm of $B_d(0)$ where where $U_c^- = U_c^+ = U^\pm$, but it is absurd since some $\check{S}_i^\pm \subset S^\pm$ where $U_{c_i}^- = U_{c_i}^+ = U^\pm$ holds for some $i \in (1, 2, 3, 4)$. For each $x \in \check{S}_i^\pm$, $(x, v = \partial_y \bar{G}(x, \partial U^\pm(x)))$ determines an orbit of the Lagrangian flow which approaches both to the support of μ_{c_i} and to the support of μ_c as $t \rightarrow \pm\infty$, it is impossible.

5.2. The Mañé set for $c \in \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$. What remains to consider is when $c \in \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$. Let us note that it contains at most countably many vertexes. Indeed, if both $\partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ and $\partial^* \mathbb{F}_0$ are non-empty, there do exist countably many vertexes (cf. Theorem 3.3).

Let $E_i \subset \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ be a sub-flat joined to other two sub-flats at the vertex c_i, c_{i+1} respectively. By Theorem 3.3, the Aubry set for c_j consists of two minimal homoclinic curves γ_{j-1} and γ_j . Denote by $g_j \in \mathbb{Z}^2$ the homology class of γ_j , then the matrix (g_{j-1}, g_j) is uni-module. By introducing suitable coordinates on \mathbb{T}^2 , we can assume $g_i = (1, 0)$. In this coordinate system, $g_{i-1} = (k, 1)$ and $g_{i+1} = (k', -1)$.

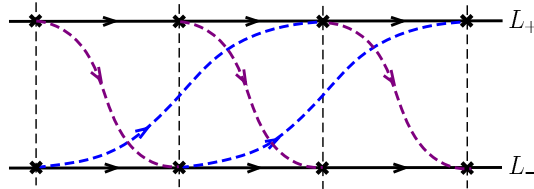


FIGURE 7.

In this figure, each unit square represents a fundamental domain of \mathbb{T}^2 in the universal covering space, the horizontal line represents the lift of the homoclinic curve γ_i , which stays in the Aubry set for each $c \in E_i$. The blue dashed lines represent the lift of the γ_{i-1} which stays in the Aubry set for the class at one end-point of E_i . The purple dashed lines represents the lift of the γ_{i+1} which stays in the Aubry set for the class at another end-point of E_i .

Let us consider weak KAM solution $U_{i,\pm}^\pm$ in the strip bounded by the lines L_+ and L_- , $U_{i,\pm}^-$ determines backward semi-static curves approaching to the line L_\pm as the time approaches to minus infinity, $U_{i,\pm}^+$ determines forward semi-static curves approaching to the line L_\mp as the time approaches to positive infinity respectively. These functions are independent of $c \in E_i$. Indeed, if they depends on c , there would be $c, c' \in E_i$ and two forward semi-static curves $\xi_c, \xi_{c'}: \mathbb{R}_+ \rightarrow \mathbb{R}^2$ such that $\xi_c(0) = \xi_{c'}(0)$ and

$$\begin{aligned} A_L(\xi_c, \dot{\xi}_c) - \langle \xi_c(\infty) - \xi_c(0), c' \rangle &> A_L(\xi_{c'}, \dot{\xi}_{c'}) - \langle \xi_{c'}(\infty) - \xi_{c'}(0), c' \rangle, \\ A_L(\xi_c, \dot{\xi}_c) - \langle \xi_c(\infty) - \xi_c(0), c \rangle &< A_L(\xi_{c'}, \dot{\xi}_{c'}) - \langle \xi_{c'}(\infty) - \xi_{c'}(0), c \rangle. \end{aligned}$$

It induces $\langle \xi_c(\infty) - \xi_{c'}(\infty), c - c' \rangle > 0$. But on the other hand $\langle g_i, c - c' \rangle = 0$ holds for any $c, c' \in E_i$, the contradiction verifies our claim.

If we remove the coercive condition on these weak KAM solutions that $(x, \partial U^\pm(x))$ determines a backward (forward) semi-static curve which must approach to L_+ (L_-), then the weak KAM depends on $c \in E_i$. In other words, let us consider the weak KAM on cylinder $\mathbb{R} \times \mathbb{T}$. Indeed, for each $c \in \text{int} E_i$, we have $A_c(\gamma_{i\pm 1}) > 0$. Thus, starting from a point very close to the line L_+ (L_-), the backward (forward) semi-static curve will approach to L_+ (L_-).

Let $c_\lambda = \lambda c_i + (1 - \lambda)c_{i+1}$. For each $\lambda \in (0, 1)$, the strip is divided into two parts D_λ^+ and D_λ^- such that $U_{c_\lambda}^\pm|_{D_\lambda^+} = U_{i,+}^\pm$ and $U_{c_\lambda}^\pm|_{D_\lambda^-} = U_{i,-}^\pm$. Let $\gamma_{x,\pm}^+$ be the forward semi-static curve determined by $U_{i,\pm}^-$, starting from x . If $x \in D_\lambda^+$, then

$$A_L(\gamma_{x,+}^+) - \langle \gamma_{x,+}^+(\infty) - x, c_\lambda \rangle < A_L(\gamma_{x,-}^+) - \langle \gamma_{x,-}^+(\infty) - x, c_\lambda \rangle.$$

For $\lambda' > \lambda$, we have

$$\langle \gamma_{x,+}^+(\infty) - \gamma_{x,-}^+(\infty), c_{\lambda'} - c_\lambda \rangle > 0.$$

It implies $x \in D_{\lambda'}^+$ also. Clearly, D_λ^+ expands and D_λ^- shrinks as λ increases. As the limit, we see that D_1^- and D_0^+ occupies the whole strip. Therefore, we have:

Proposition 5.1. *Assume $E_i \subset \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$ be a sub-flat joined to other two sub-flats at the vertex c_i, c_{i+1} respectively. Let U_j^\pm be the globally elementary weak KAM for $c_\lambda = c_j$ with $j = i, i+1$. Then, for each $c = \lambda c_i + (1 - \lambda)c_{i+1} \in E_i$, the weak KAM solution is totally determined by U_i^\pm and U_{i+1}^\pm in the following sense: there is a partition $\mathbb{T}^2 = D_{i,\lambda}^\pm \cup D_{i+1,\lambda}^\pm$ such that*

$$U_{c_\lambda}^\pm|_{D_{j,\lambda}^\pm} = U_j^\pm|_{D_{j,\lambda}^\pm},$$

where $D_{i,\lambda}^\pm \subset D_{i,\lambda'}^\pm$ and $D_{i+1,\lambda}^\pm \supset D_{i+1,\lambda'}^\pm$ if $\lambda < \lambda'$, and $D_{i+1,0}^\pm = D_{i,1}^\pm = \mathbb{T}^2$.

In virtue of this lemma, we see that, for all $c \in \partial \mathbb{F}_0 \setminus \partial^* \mathbb{F}_0$, the set of barrier functions $\{U_c - U_c^+\}$ are determined by countably many weak KAM solutions. For each sub-flat E_i , there is an open-dense set in C^r space such that if the potential V takes value in this set, then $\arg \min(U_c - U_c^+) \subsetneq \mathbb{T}^2$ holds for each $c \in E_i$.

Theorem 5.1. *Let $\bar{L} = \frac{1}{2} \langle A^{-1} \dot{x}, \dot{x} \rangle - V(x)$. A residual set $\mathfrak{V} \subset C^r(\mathbb{T}^2, \mathbb{R})$ exists such that for each $V \in \mathfrak{V}$ the following holds: for each $c \in \partial \mathbb{F}_0$, the Mañé set does not cover the whole configuration space: $\arg \min(U_c^- - U_c^+) \subsetneq \mathbb{T}^2$.*

Therefore, the following is also a generic property:

H5: The potential V is chosen so that the Mañé set does not cover the whole torus $\mathcal{N}(c) \subsetneq \mathbb{T}^2$ for each $c \in \partial\mathbb{F}_0$.

5.3. Thickness of the annulus. The Hamiltonian we study is given by (4.1). To start with, let us study elementary weak KAM solutions of the Hamilton-Jacobi equation

$$(5.1) \quad \partial_\tau u + G(x, \tau, \partial_x u + c) = \epsilon \Delta, \quad c \in \partial\mathbb{F}_0$$

by assuming $\min \alpha = 0$. Where G solves the equation $H(x, x_3, y, G) = E$ and $\tau = -x_3$. Expanding the Hamiltonian into Taylor series of ϵ , replacing u by $\sqrt{\epsilon}u$ and rescaling τ by $s = \sqrt{\epsilon}\tau$ we obtain the equation

$$(5.2) \quad \frac{\partial u}{\partial s} + \frac{1}{2} \left\langle A \left(\frac{\partial u}{\partial x} + c \right), \frac{\partial u}{\partial x} + c \right\rangle + V(x) + O(\sqrt{\epsilon}) = \Delta$$

where A is the Hessian matrix of h in y at $y = 0$, $V(x) = Z(x, 0)$. In this equation, only higher order term depends on the time s . As the first step to study the weak KAM of this Hamilton-Jacobi equation, we omit the higher order term and let $\Delta = 0$. Since the potential V is independent of the time s , all weak KAM solutions solve the homogenized equation

$$(5.3) \quad \frac{1}{2} \left\langle A \frac{\partial u}{\partial x} + c, \frac{\partial u}{\partial x} + c \right\rangle + V(x) = \Delta$$

with $\Delta = 0$. According to the study in the last two sub-sections, for each $V \in \mathfrak{V} \subset C^r(\mathbb{T}^2, \mathbb{R})$ and each $c \in \partial\mathbb{F}_0$, the weak KAM solutions of this equation define a Mañé set which does not cover the whole torus: $\arg \min(U_c^- - U_c'^+) \subsetneq \mathbb{T}^2$. By the upper semi-continuity of the set of semi-static curves, some small $\Delta_0 > 0$ exists such that for each positive $\Delta \leq \Delta_0$ and each $c \in \alpha_G^{-1}(\Delta)$ (We use $\alpha_{\bar{G}}$ to denote the α -function for the Lagrangian determined by the Hamiltonian \bar{G}) the Mañé set does not cover the torus. It implies that $\arg \min(U_c^- - U_c'^+) \subsetneq \mathbb{T}^2$ holds if $c \in \alpha_{\bar{G}}(\Delta)$, both U_c^- and $U_c'^+$ are the weak KAM solutions of the equation 5.3 with $\Delta \leq \Delta_0$.

For each average action $\Delta > \min \alpha_{\bar{G}}$, the dynamics on the energy level $\bar{G}^{-1}(\Delta)$ is similar to twist and area-preserving maps. First of all, the rotation vector of each minimal measure is not zero, consequently, it is not supported on fixed points. Second, for each class $c \in \alpha_{\bar{G}}^{-1}(\Delta)$, all c -minimal measures share the same rotation direction, otherwise, the Lipschitz graph property will be violated. If the rotation direction is rational, the minimal measure is supported on periodic orbit.

From these properties one can see the following: for each $c \in \alpha_{\bar{G}}^{-1}(\Delta)$, there exists a circle $\Gamma_c \subset \mathbb{T}^2$ such that each semi-static curve passes through it transversally and $\arg \min(U_c^- - U_c'^+) \cap \Gamma_c \subsetneq \Gamma_c$. Since the set $\arg \min(U_c^- - U_c'^+)$ is closed, there exist finitely many intervals $I_i \subset \Gamma_c$ disjoint to each other such that $(\arg \min(U_c^- - U_c'^+) \cap \Gamma_c) \subset \cup I_i$.

As these functions are independent of $s = \sqrt{\epsilon}\tau$, all of these functions can be thought as the weak KAM solutions of the homogenized Hamilton-Jacobi equation

$$\frac{\partial u}{\partial s} + \frac{1}{2} \left\langle A \left(\frac{\partial u}{\partial x} + c \right), \frac{\partial u}{\partial x} + c \right\rangle + V(x) = \Delta$$

if they are thought as the function of the variable (x, s) . Here, the cohomology class takes value on the circle: $c \in \alpha_{\bar{G}}^{-1}(\Delta)$. It follows that, for each class $c \in \alpha_{\bar{G}}^{-1}(\Delta)$, there exists non-degenerate embedded two-torus $\Gamma_c \times \mathbb{T} \subset \mathbb{T}^3$ and finitely many intervals

$I_i \subset \Gamma_c$ disjoint to each other such that each c -semi-static curve passes through the two-torus transversally and $\arg \min(U_c^- - U_c^+) \cap \Gamma_c \times \mathbb{T} \subset \cup I_i \times \mathbb{T}$. Here the circle \mathbb{T} is for the time $s = \sqrt{\epsilon}\tau$.

Let us come back to the Hamilton-Jacobi equation (5.2). Recall the normal form of the Hamiltonian, we see that in the remainder $O(\sqrt{\epsilon})$, one contribution is from $R'(x, \sqrt{\epsilon}p, \sqrt{\epsilon}^{-1}s)$, other contributions are independent of τ . Again, by the upper semi-continuity of the set of semi-static curves, we have

Theorem 5.2. *Under the hypotheses (H1~H3, H5), some positive numbers $\Delta_0 > 0$ and $\epsilon_0 > 0$ exist, depending on the potential V , such that for each $\Delta < \Delta_0$, each $\epsilon < \epsilon_0$ and each $c \in \alpha^{-1}(\Delta)$, all semi-static curves pass through transversally the two-torus $\Gamma_c \times \{s \in \mathbb{T}\}$,*

$$(5.4) \quad \arg \min(U_c^- - U_c^+) \cap (\Gamma_c \times \{s = \text{const.}\}) \subset \bigcup I_{c,i}$$

where Γ_c is a circle restricted in a 2-torus $\{s = \text{constant}\} \subset \mathbb{T}^3$, $I_{c,i} \subset \Gamma_c$ are closed intervals, disjoint to each other and independent of the time s .

Here, the semi-static curves are in the sense of extended configuration space, i.e. if $\gamma : \mathbb{R} \rightarrow M$ is a curve, we call its graph a curve also $\tilde{\gamma}(s) = (\gamma(s), s) \in M \times \mathbb{T}$.

Let us go back to the original scale. By Theorem 5.2, there exist a annulus-shaped region

$$\mathbb{A} = \{c : 0 < \alpha_G(c) - \min \alpha_G \leq \epsilon \Delta_0\}$$

such that Condition 5.4 holds for each $c \in \mathbb{A}$. Recall this time-periodic system is deduced from the autonomous system restricted on certain energy level $\bar{H}^{-1}(E)$. In virtue of Theorem 3.4, the counterpart of \mathbb{A} in $H^1(\mathbb{T}^3, \mathbb{R})$ is

$$\mathbb{A}' = \{\tilde{c} = (c, c_3) \in \alpha^{-1}(E) : 0 < c_3 \leq \epsilon \Delta_0\},$$

where we notice that the sphere $\alpha^{-1}(E)$ is located in the upper half space of \mathbb{R}^3 and touches the plane $\{c_3 = 0\}$ where $c \in \mathbb{F}_0$.

In the original coordinates $(\tilde{x}, \tilde{y}) = (x, x_3, y, y_3)$, Theorem 5.2 states such a fact: for each $\tilde{c} \in \mathbb{A}$, all \tilde{c} -semi static curves pass through the 2-torus $\Gamma_c \times \{x_3 \in \mathbb{T}\}$ transversally and all intersection points are restricted in the strips $\cup I_{c,i} \times \{x_3 \in \mathbb{T}\}$. However, the condition 5.4 do not guarantee complete “intersection” of the stable set with unstable set, in that case, the set $\arg \min(U_c^- - U_c^+) \cap \{\tau = 0\}$ contains some disconnected points. Therefore, we call \mathbb{A} the annulus of incomplete intersection. In this case, we do not expect to construct orbits connecting each Aubry set to any other Aubry set nearby, the incompleteness blocks some direction. However, once non-trivial intersection exists, it opens way to connect some Aubry set nearby. One can see it in the subsection 7.2.

As $\epsilon \Delta_0 \gg D\epsilon^{1+(r-2)\sigma}$ provided $\epsilon > 0$ is sufficiently small, we obtained

Overlap Property: *For any two irreducible $g, g' \in H_1(\mathbb{T}^2, \mathbb{Z})$, there exists $\epsilon_0 = \epsilon_0(V, g, g') > 0$ such that the wedge-shaped regions intersects the annulus-shaped region: $\mathbb{W} \cap \mathbb{A} \neq \emptyset$ and $\mathbb{W}' \cap \mathbb{A} \neq \emptyset$ provided $0 < \epsilon \leq \epsilon_0$.*

6. LOCAL CONNECTING ORBITS

To construct orbits connecting some Aubry set to another Aubry set nearby, we introduce two types of modified Tonelli Lagrangian, i.e. the time-step and the space-step Lagrangian. They satisfy the conditions of the positive definiteness, the super-linear growth and the completeness. The time-step Lagrangian $L : TM \times \mathbb{R} \rightarrow \mathbb{R}$ is not periodic in t on the whole \mathbb{R} , instead, it is periodic when it is restricted either on $(-\infty, -\delta)$ or on (δ, ∞) , namely, $L(\cdot, t) = L(\cdot, t+1)$ if $t, t+1 \in (-\infty, -\delta)$ or $t, t+1 \in (\delta, \infty)$. The second type of Lagrangian is defined on some covering space. Let $\pi_1 : \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} \rightarrow M$. The space-step Lagrangian $L : T\bar{M} \times \mathbb{T} \rightarrow \mathbb{R}$ is not periodic in one component of spaces coordinates x_1 . It is periodic in x_1 when it is restricted either on $(-\infty, -\delta)$ or on (δ, ∞) , namely, $L(x_1, \cdot) = L(x_1 + 1, \cdot)$ if $x_1, x_1 + 1 \in (-\infty, -\delta)$ or $x_1, x_1 + 1 \in (\delta, \infty)$.

The existence of local connecting orbits is established based on some upper semi-continuity of minimal curves for the modified Lagrangian.

6.1. Upper semi-continuity of minimal curves. Time-step Lagrangian: Let us consider time-step Lagrangian first. A curve $\gamma : \mathbb{R} \rightarrow M$ is called minimal if

$$\int_{\tau}^{\tau'} L(\gamma(t), \dot{\gamma}(t), t) dt \leq \int_{\tau}^{\tau'} L(\zeta(t), \dot{\zeta}(t), t) dt$$

holds for any $\tau < \tau'$ and for any absolutely continuous curve $\zeta : [\tau, \tau'] \rightarrow M$ with $\zeta(\tau) = \gamma(\tau)$ and $\zeta(\tau') = \gamma(\tau')$. Let $\mathcal{G}(L)$ denote the set of minimal curves for L . Let $\tilde{\mathcal{G}}(L) = \bigcup_{\gamma \in \mathcal{G}(L)} (\gamma(t), \dot{\gamma}(t), t)$, $\mathcal{G}(L) = \pi \tilde{\mathcal{G}}(L)$ where $\pi : TM \times \mathbb{R} \rightarrow M \times \mathbb{R}$ is the standard projection.

Theorem 6.1. *The set-valued map $L \rightarrow \mathcal{G}(L)$ is upper semi-continuous. Consequently, the map $L \rightarrow \tilde{\mathcal{G}}(L)$ is also upper semi-continuous.*

Proof. Let K be the diameter of the closed manifold M , namely,

$$K = \max_{x, x' \in M} \ell(x, x')$$

where $\ell(x, x')$ denotes the length of the shortest geodesic connecting x with x' . Let

$$K_1 = \sup_{\substack{(x, t) \in M \times \mathbb{R} \\ \|v\| \leq K}} L(x, v, t).$$

As L is assumed periodic for $t \leq 0$ as well as for $t \geq 1$, K_1 is finite.

Given time interval $[\tau, \tau']$ with $\tau' - \tau \geq 1$, we re-parameterize the shortest geodesic $\gamma(s)$ by $\gamma'(\ell(x, x')(t - \tau)/(\tau' - \tau))$, then $\gamma' : [\tau, \tau'] \rightarrow M$ is C^1 -curve such that $\gamma'(a) = x$, $\gamma'(a) = x'$. Clearly, the action along this curve is not bigger than $K_1(\tau' - \tau)$. Obviously, there is an upper bound uniformly for all minimizing action of L' if it is close to L on $\{\|v\| \leq K\}$, still denoted by

$$h_{L'}((x, \tau), (x', \tau')) \leq K_1(\tau' - \tau).$$

If the Lagrangian is assumed super-linear growth, some positive numbers $C, D > 0$ exist such that $L'(x, \dot{x}, t) \geq C\|\dot{x}\| - D$ for all $(x, \dot{x}, t) \in TM \times \mathbb{R}$ and for all L' close

to L . Thus, one obtains

$$(6.1) \quad \frac{\text{dist}(\gamma(\tau), \gamma(\tau'))}{\tau' - \tau} \leq \frac{1}{\tau' - \tau} \int_{\tau}^{\tau'} \|d\gamma\| \leq \frac{K + D}{C}$$

if γ is a minimizer. As (6.1) holds for any $\tau' - \tau \geq 1$, it implies that there must be some $t_i \in [\tau + i, \tau + i + 1]$ ($i \in \mathbb{Z}$) such that $\|\dot{\gamma}(t_i)\| \leq C^{-1}(K_1 + D)$. As it holds for any $x, x' \in M$, therefore, $K_2 > 0$ exists such that

$$\phi^s \left(\left\{ x, v, t_i : \|v\| \leq \frac{K + D}{C} \right\} \right) \subset \left\{ x, v, t_i + s : \|v\| \leq K_2 \right\}$$

holds for all $s \in [0, 2]$ and for all relevant i . It implies that $\|\dot{\gamma}(t)\| \leq K_2$ holds for all $t \in [\tau, \tau']$.

Let $L_i \in C^r(TM \times \mathbb{R}, \mathbb{R})$ be a sequence converging to L in the following sense: there exists some $U_k \supset \{x, v, t : \|v\| \leq K_2\}$, as well as a sequence of $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$ such that $\|L - L_i\|_{C^2(U_k, \mathbb{R})} \leq \epsilon_i$. Let $\gamma_i: [\tau, \tau'] \rightarrow M$ be the minimizer of L_i with $\tau' - \tau \geq 1$, we then have $\|\dot{\gamma}_i(t)\| \leq K_2$ for all $t \in [\tau, \tau']$. The set $\{\gamma_i\}$ is compact in the $C^1([\tau, \tau'], M)$ -topology. Indeed, since $\partial^2 L / \partial \dot{x}^2$ is positive definite one can write the Lagrange equations in the form of $\ddot{x} = f(x, \dot{x}, t)$, which implies γ_i is bounded in C^2 -topology.

Let $\gamma: [\tau, \tau'] \rightarrow M$ be one of the accumulation points of this set. Clearly, $\gamma: [\tau, \tau'] \rightarrow M$ is the minimizer of L . Let $I_i = [\tau_i, \tau'_i]$ and let $\tau_i \rightarrow -\infty$ and $\tau'_i \rightarrow \infty$, we obtain a sequence of minimizers of L_i , $\gamma_i: I_i \rightarrow M$. By diagonal extraction argument some subsequence of γ_i which converges C^1 -uniformly on each compact set to a C^1 -curve $\gamma: \mathbb{R} \rightarrow M$ which is the minimizer of L on any compact interval of \mathbb{R} . This proves the upper semi-continuity. \square

In application, the set $\mathcal{G}(L)$ seems too big for the construction of connecting orbits. For time-periodic Lagrangian, Mañé set can be a proper subset of $\tilde{\mathcal{G}}(L)$, $\tilde{\mathcal{N}}(L) \subsetneq \tilde{\mathcal{G}}(c)$. It is closely related to the problem whether the Lax-Oleinik semi-group converges or not (cf. [FM]). For time-step Lagrangian, pseudo connecting curve is introduced to play roles similar to what semi-static curve does.

Each time-step Lagrangian L uniquely determines two time-periodic Lagrangian L^+ and L^- such that $L^+|_{(\delta, \infty)} = L|_{(\delta, \infty)}$ and $L^-|_{(-\infty, -\delta)} = L|_{(-\infty, -\delta)}$. Let $-\alpha^\pm$ denote the minimal average action of L^\pm . For $m_0, m_1 \in M$, we define

$$h_L^{T_0, T_1}(m_0, m_1) = \inf_{\substack{\gamma(-T_0) = m_0 \\ \gamma(T_1) = m_1}} \int_{-T_0}^{T_1} L(d\gamma(t), t) dt + T_0 \alpha^- + T_1 \alpha^+.$$

Clearly the limit infimum is bounded

$$|h_L^\infty(m_0, m_1)| = \liminf_{T_0, T_1 \rightarrow \infty} |h_L^{T_0, T_1}(m_0, m_1)| < \infty.$$

Let $\{T_0^i\}_{i \in \mathbb{Z}_+}$ and $\{T_1^i\}_{i \in \mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i \rightarrow \infty$ ($j = 0, 1$) as $i \rightarrow \infty$ and the following limit exists

$$\lim_{i \rightarrow \infty} h_L^{T_0^i, T_1^i}(m_0, m_1) = h_L^\infty(m_0, m_1).$$

Let $\gamma_i(t, m_0, m_1): [-T_0^i, T_1^i] \rightarrow M$ be a minimizer connecting m_0 and m_1

$$h_L^{T_0^i, T_1^i}(m_0, m_1) = \int_{-T_0^i}^{T_1^i} L(d\gamma_i(t), t) dt + T_0^i \alpha^- + T_1^i \alpha^+.$$

From the proof of Theorem 6.1 one can see that for any compact interval $[a, b]$ there is some $I \in \mathbb{Z}_+$ such that the set $\{\gamma_i\}_{i \geq I}$ is pre-compact in $C^1([a, b], M)$.

Lemma 6.1. *Let $\gamma: \mathbb{R} \rightarrow M$ be an accumulation point of $\{\gamma_i\}$. If $s, \tau \geq \delta$ then*

$$(6.2) \quad \begin{aligned} A_L(\gamma|[-s, \tau]) &= \inf_{\substack{s_1 - s \in \mathbb{Z}, \tau_1 - \tau \in \mathbb{Z} \\ s_1, \tau_1 \geq \delta \\ \gamma^*(-s_1) = \gamma(-s) \\ \gamma^*(\tau_1) = \gamma(\tau)}} \int_{-s_1}^{\tau_1} L(d\gamma^*(t), t) dt \\ &\quad + (s_1 - s)\alpha^- + (\tau_1 - \tau)\alpha^+. \end{aligned}$$

Proof. : To prove the lemma let us suppose the contrary. Thus there would exist $\Delta > 0$, $s_1, \tau_1 \geq \delta$, $s_1 - s \in \mathbb{Z}$, $\tau_1 - \tau \in \mathbb{Z}$ and a curve $\gamma^*: [s_1, \tau_1] \rightarrow M$ with $\gamma^*(-s_1) = \gamma(-s)$, $\gamma^*(\tau_1) = \gamma(\tau)$ such that

$$A_L(\gamma|[s, \tau]) \geq \int_{-s_1}^{\tau_1} L(d\gamma^*(t), t) dt + (s_1 - s)\alpha^- + (\tau_1 - \tau)\alpha^+ + \Delta.$$

Let $\epsilon = \frac{1}{4}\Delta$. By the definition of limit infimum there exist $T_0^{i_0} > s$ and $T_1^{i_0} > \tau$ such that

$$h_L^{T_0, T_1}(m_0, m_1) \geq h_L^\infty(m_0, m_1) - \epsilon, \quad \forall \quad T_0 \geq T_0^{i_0}, \quad T_1 \geq T_1^{i_0},$$

there exist subsequences $T_j^{i_k}$ ($j = 0, 1; k = 0, 1, 2, \dots$) such that $T_0^{i_k} - T_0^{i_0} \geq |s - s_1|$, $T_1^{i_k} - T_1^{i_0} \geq |\tau - \tau_1|$ and

$$|h_L^{T_0^{i_k}, T_1^{i_k}}(m_0, m_1) - h_L^\infty(m_0, m_1)| < \epsilon$$

holds for each $k > 0$. Let γ_{i_k} be the minimizer of $h_L^{T_0^{i_k}, T_1^{i_k}}(m_0, m_1)$. By taking a subsequence further one can assume $\gamma_{i_k} \rightarrow \gamma$. In this case, for sufficiently large k we can construct a curve $\gamma_{i_k}^*: [s_1, \tau_1] \rightarrow M$ which has the same endpoints as γ_{i_k} : $\gamma_{i_k}^*(s_1) = \gamma_{i_k}(s)$, $\gamma_{i_k}^*(\tau_1) = \gamma_{i_k}(\tau)$ and satisfies the following

$$A_L(\gamma_{i_k}|[s, \tau]) \geq \int_{s_1}^{\tau_1} L(d\gamma_{i_k}^*(t), t) dt + (s_1 - s)\alpha^- + (\tau_1 - \tau)\alpha^+ + \frac{3}{4}\Delta.$$

Extending $\gamma_{i_k}^*$ from $[s_1, \tau_1]$ to the $[-T_0^{i_k} - (s_1 - s), T_1^{i_k} + (\tau_1 - \tau)]$ by

$$\gamma_{i_k}^* = \begin{cases} \gamma_{i_k}(t + s_1 - s), & t \leq -s_1, \\ \gamma_{i_k}^*(t), & -s_1 \leq t \leq \tau_1, \\ \gamma_{i_k}(t - \tau_1 + \tau), & t \geq \tau_1, \end{cases}$$

and defining $T_0' = T_0^{i_k} + (s_1 - s)$, $T_1' = T_1^{i_k} + (\tau_1 - \tau)$ we find that

$$\begin{aligned} h_L^{T_0', T_1'}(m_0, m_1) &\leq A_L(\gamma_{i_k}^*|[-T_0', T_1']) + T_0' \alpha^- + T_1' \alpha^+ \\ &\leq A_L(\gamma_{i_k}|[-T_0^{i_k}, T_1^{i_k}]) + T_0^{i_k} \alpha^- + T_1^{i_k} \alpha^+ - \frac{3}{4}\Delta \\ &\leq h_L^\infty(m_0, m_1) - 2\epsilon. \end{aligned}$$

But this contradicts the definition of the limit infimum as $T_0' \geq T_0$ and $T_1' \geq T_1$. \square

We define so-called pseudo connecting curve set

$$\mathcal{C}(L) = \{\gamma \in \mathcal{G}(L) : (6.2) \text{ hold}\}.$$

In application, we usually choose time-step Lagrangian so that the Aubry set of L^- is different from that of L^+ . Clearly, for $\gamma \in \mathcal{C}(L)$, the orbit $(\gamma(t), \dot{\gamma}(t))$ approaches to the Aubry set $\tilde{\mathcal{A}}(L^-)$ as $t \rightarrow -\infty$ and approaches to $\tilde{\mathcal{A}}(L^+)$ as $t \rightarrow \infty$. That is why we call it pseudo connecting curve. Let

$$\tilde{\mathcal{C}}(L) = \bigcup_{\gamma \in \mathcal{C}(L)} (\gamma(t), \dot{\gamma}(t), t), \quad \mathcal{C}(L) = \bigcup_{\gamma \in \mathcal{C}(L)} (\gamma(t), t).$$

Clearly, if L is periodic in t , then $\tilde{\mathcal{C}}(L) = \tilde{\mathcal{N}}(L)$ and $\mathcal{C}(L) = \mathcal{N}(L)$.

Theorem 6.2. *The map $L \rightarrow \mathcal{C}(L)$ is upper semi-continuous.*

Proof. : Let $L_i \rightarrow L$ be a sequence of time-step Lagrangian, let $\gamma_i \in \mathcal{C}(L_i)$ and let γ be an accumulation point of the set $\{\gamma_i \in \mathcal{C}(L_i)\}_{i \in \mathbb{Z}^+}$. We claim that $\gamma \in \mathcal{C}(L)$. If $\gamma \notin \mathcal{C}(L)$, there would be two point $\gamma(s), \gamma(\tau) \in M$ connected by another curve $\gamma^*: [s + n_1, \tau + n_2] \rightarrow M$ and $\Delta > 0$ such that

$$A_L(\gamma|_{[s, \tau]}) < A_L(\gamma^*) + n_1\alpha^- + n_2\alpha^+ - \Delta$$

where $s, s + n_1 \leq -\delta$, $\tau, \tau + n_2 \geq \delta$. Since γ is an accumulation point of γ_i , for any small $\epsilon > 0$, there would be sufficiently large i such that $\|\gamma - \gamma_i\|_{C^1[s, t]} < \epsilon$, and above inequality also holds for $A_{L_i}(\gamma_i|_{[s, \tau]})$. It follows that $\gamma_i \notin \mathcal{C}(L_i)$ but that is absurd. \square

As an immediate consequence, the map $c \rightarrow \tilde{\mathcal{N}}(c)$ as well as the map $c \rightarrow \mathcal{N}(c)$ is upper semi-continuous.

Space-step Lagrangian: Let $M = \mathbb{T}^n$ and $\pi_1 : \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} \rightarrow M$, where \mathbb{R} is for the coordinate x_1 . The space-step Lagrangian L is introduced to handle the problem of incomplete intersection. It also uniquely determines two Lagrangian L^- and L^+ : TM such that $L^-(x_1, \cdot)|_{(-\infty, -\delta)} = L(x_1, \cdot)|_{(-\infty, -\delta)}$ and $L^+(x_1, \cdot)|_{(\delta, \infty)} = L(x_1, \cdot)|_{(\delta, \infty)}$ if we treat L^\pm as its natural lift to $T\bar{M}$. Let μ^\pm denote minimal measure of L^\pm with 0-cohomology class, $\omega(\mu^\pm) = (\omega_1(\mu^\pm), \dots, \omega_n(\mu^\pm))$ denote the rotation vector. We assume some conditions on the Lagrangian:

- 1, $\omega_1(\mu^\pm) > 0$ for each ergodic minimal measure μ^\pm ;
- 2, $\alpha_{L^-}(0) = \alpha_{L^+}(0)$, without losing of generality, it equals zero.
- 3, $|L^- - L^+| \leq \frac{1}{2} \min_{\omega_1=0} \{\beta_{L^-}(\omega'), \beta_{L^+}(\omega')\}$.

It is shown in [Lx] that some coordinates exists such that the first condition holds provided $\alpha(0) > \min \alpha$. The third condition makes sense because $\omega_1(\mu^\pm) > 0$ implies $\inf_{\omega_1(\mu)=0} \int L^\pm d\mu > \inf \int L^\pm d\mu$. To introduce minimal curve for space-Lagrangian, we define

$$h_L^T(\bar{m}_0, \bar{m}_1) = \inf_{\substack{\bar{\gamma}(-T)=\bar{m}_0 \\ \bar{\gamma}(T)=\bar{m}_1}} \int_{-T}^T L(\bar{\gamma}(t), \dot{\bar{\gamma}}(t)) dt.$$

where $\bar{m}_0, \bar{m}_1 \in \bar{M}$.

Lemma 6.2. *If the rotation vector of each ergodic minimal measure has positive first component $\omega_1(\mu^\pm) > 0$, $\bar{m}_0 \neq \bar{m}_1$, then*

$$\lim_{T \rightarrow 0} h_L^T(\bar{m}_0, \bar{m}_1) = \infty \quad \text{and} \quad \lim_{T \rightarrow \infty} h_L^T(\bar{m}_0, \bar{m}_1) = \infty.$$

Proof. The case for $T \rightarrow 0$ is an immediate consequence of the super-linear growth of L in \dot{x} .

Let $\bar{\gamma}_L^T: [-T, T] \rightarrow \bar{M}$ be the minimizer of $h_L^T(\bar{m}_0, \bar{m}_1)$. Let $m_0 = \pi_1 \bar{m}_0$, $m_1 = \pi_1 \bar{m}_1$, $\zeta: [0, 1] \rightarrow M$ be a smooth curve connecting m_1 to m_0 , $\dot{\zeta}(0) = \dot{\gamma}_L^T(T)$ and $\dot{\zeta}(1) = \dot{\gamma}_L^T(-T)$. The action of L^+ along ζ is clearly bounded, thus for any $\epsilon > 0$, one has $A_{L^+}(\zeta) \leq 2T\epsilon$ provided T is sufficiently large. The curve $\xi = \zeta * \pi_1 \bar{\gamma}_L^T$ uniquely determines a holonomic probability measure $\nu_L^T \in \mathfrak{H}$ such that

$$\int f d\nu_L^T = \frac{1}{2T+1} \int_{-T}^{T+1} f(\xi(t), \dot{\xi}(t)) dt$$

holds for each $f \in C(TM \times \mathbb{T})$. Clearly, $\omega_1(\nu_L^T) \rightarrow 0$ as $T \rightarrow \infty$.

$$\begin{aligned} \frac{1}{2T} h_L^T(\bar{m}_0, \bar{m}_1) &= \frac{2T+1}{2T} \int L^+ d\nu_L^T - \frac{1}{2T} \int_0^1 L^+(\zeta(t), \dot{\zeta}(t)) dt \\ &\quad + \frac{1}{2T} \int_{-T}^T (L - L^+)(\bar{\gamma}_L^T(t), \dot{\gamma}_L^T(t)) dt \\ &\geq \int L^+ d\nu_L^T - \frac{1}{2} \min_{\omega_1=0} \beta_{L^+}(\omega) - \epsilon \\ &\geq \frac{1}{2} \min_{\omega_1=0} \beta_{L^+}(\omega) > 0. \end{aligned}$$

It follows that $\lim_{T \rightarrow \infty} h_L^T(\bar{m}_0, \bar{m}_1) = \infty$. □

As an intermediate step in introducing pseudo-connecting curve, we define a set of minimal curve $\mathcal{G}(L)$.

Definition 6.1. *A curve $\bar{\gamma}: \mathbb{R} \rightarrow \bar{M}$ is in $\mathcal{G}(L)$ if*

$$A_L(\bar{\gamma}|_{[-T, T]}) = \inf_{T' \in \mathbb{R}_+} h_L^{T'}(\bar{\gamma}(-T), \bar{\gamma}(T)).$$

We claim that $\mathcal{G}(L) \neq \emptyset$. Indeed, denote by $\bar{\gamma}_L(\cdot, \bar{m}_0, \bar{m}_1): [-T, T] \rightarrow M$ the minimizer such that $\bar{\gamma}_L(-T) = \bar{m}_0$, $\bar{\gamma}_L(T) = \bar{m}_1$ and

$$A(\bar{\gamma}_L) = \int_{-T}^T L(\bar{\gamma}_L(t), \dot{\gamma}_L(t)) dt = \inf_{T' \in \mathbb{R}_+} h_L^{T'}(\bar{m}, \bar{m}').$$

Because of Lemma 6.2, this infimum is attained for finite $T > 0$ if \bar{m}_0 and \bar{m}_1 are two different points in \bar{M} . The super-linear growth of L in \dot{x} guarantees that $T \rightarrow \infty$ as $-\bar{m}_{01}, \bar{m}_{11} \rightarrow \infty$, where \bar{m}_{i1} denotes the first coordinate of \bar{m}_i . Given an interval $[-T, T]$, for sufficiently large $-\bar{m}_{01}, \bar{m}_{11}$, the set $\{\bar{\gamma}_L(\cdot, \bar{m}_0, \bar{m}_1)|_{[-T, T]}\}$ is pre-compact in $C^1([-T, T], \bar{M})$. Let $T \rightarrow \infty$. By diagonal extraction argument, there is a subsequence of $\{\bar{\gamma}_L(\cdot, \bar{m}_0, \bar{m}_1)\}$ which converges C^1 -uniformly on each compact set to a C^1 -curve $\bar{\gamma}: \mathbb{R} \rightarrow \bar{M}$. Obviously, $\bar{\gamma} \in \mathcal{G}(L)$, and

Proposition 6.1. *Some number $A > 0$ exists so that $|h_L^T(\bar{\gamma}(-T), \bar{\gamma}(T))| \leq A$ holds for each curve $\bar{\gamma} \in \mathcal{G}(L)$ and each $T > 0$.*

Proof. Since $\alpha_{L^\pm}(0) = 0$, some $A' > 0$ exists such that $|A(\gamma|_I)| \leq A'$ holds for any interval $I \subset \mathbb{R}_+(\mathbb{R}_-)$ provided it is a forward (backward) semi-static curves for L^\pm . Also, some $A'' > 0$ exists such that

$$-A'' \leq \max_{\bar{x}, \bar{x}' \in \{x \in \bar{M} : |x_1| \leq 1\}} \inf_{T \geq 0} h_L^T(\bar{x}, \bar{x}') \leq A''.$$

Consequently, we claim that $A \leq 2A' + A''$.

If there exists some $\bar{\gamma} \in \mathcal{G}(L)$ and some $T > 0$ such that $h_L^T(\bar{\gamma}(-T), \bar{\gamma}(T)) > 2A' + A''$, we join $\bar{\gamma}(-T)$ to $\bar{\gamma}(T)$ by another curve $\xi = \bar{x}_-^+ * \zeta * \bar{x}_+^-$ where \bar{x}_-^+ is a lift of forward semi-static curve x_- for L_- such that $\bar{\gamma}(-T) = \bar{x}_-^+(0)$, denote by \bar{x}_- the intersection point of this curve with the section $\{\bar{x} \in \bar{M} : \bar{x}_1 = -1\}$, \bar{x}_+^- is a lift of backward semi-static curve x_+ for L_+ such that $\bar{\gamma}(T) = \bar{x}_+^-(0)$, denote by \bar{x}_+ the intersection point of this curve with the section $\{\bar{x} \in \bar{M} : \bar{x}_1 = 1\}$, ζ is a minimal curve of L that connects the point \bar{x}_- to \bar{x}_+ . Obviously, one has $A_L(\xi) \leq 2A' + A'' < h_L^T(\bar{\gamma}(-T), \bar{\gamma}(T))$, but it contradicts the definition of $\mathcal{G}(L)$. \square

Each $k \in \mathbb{Z}$ defines a Deck transformation $\mathbf{k} : \bar{M} \rightarrow \bar{M} : \mathbf{k}x = (x_1 + k, x_2, \dots, x_n)$. Let $\bar{M}_\delta^- = \{x \in \bar{M} : x_1 < -\delta\}$, $\bar{M}_\delta^+ = \{x \in \bar{M} : x_1 > \delta\}$.

Definition 6.2. A curve $\bar{\gamma} \in \mathcal{G}(L)$ is called *pseudo connecting curve* if the following holds

$$A_L(\bar{\gamma}|_{[-T, T]}) = \inf_{\substack{T' \in \mathbb{R}_+ \\ \mathbf{k}^- \bar{\gamma}(-T) \in \bar{M}_\delta^- \\ \mathbf{k}^+ \bar{\gamma}(T) \in \bar{M}_\delta^+}} h_L^{T'}(\mathbf{k}^- \bar{\gamma}(-T), \mathbf{k}^+ \bar{\gamma}(T))$$

for each $\bar{\gamma}(-T) \in \bar{M}_\delta^-$ and $\bar{\gamma}(T) \in \bar{M}_\delta^+$. Denote by $\mathcal{C}(L)$ the set of pseudo connecting curves.

Lemma 6.3. The set $\mathcal{C}(L)$ is non-empty.

Proof. Let us start with a curve $\bar{\gamma} \in \mathcal{G}(L)$. Given $d > 0$, if some interval $[t_i^-, t_i^+]$ exists such that $\mathbf{k}_i^- \bar{\gamma}(t_i^-)$ can be connected to $\mathbf{k}_i^+ \bar{\gamma}(t_i^+)$ by another curve ζ_i with smaller action

$$A_L(\gamma|_{[t_i^-, t_i^+]}) - A_L(\zeta_i) \geq d > 0,$$

then one obtain a curve $\bar{\gamma}_i = \mathbf{k}_i^- \bar{\gamma}|_{(-\infty, t_i^-]} * \zeta * \mathbf{k}_i^+ \bar{\gamma}|_{[t_i^+, \infty)}$ by one step of such surgery.

Given any $d > 0$, we claim that there are finitely many intervals $[t_i^-, t_i^+]$ with $t_i^+ \leq t_{i+1}^-$ such that $\mathbf{k}_i^- \bar{\gamma}(t_i^-)$ can be connected to $\mathbf{k}_i^+ \bar{\gamma}(t_i^+)$ by another curve ζ_i with the action d smaller than the original one. Let us assume the contrary. Then, for any positive integer k , there is an interval $[-T, T] \supset \cup_{i=1}^k [t_i^-, t_i^+]$. Without losing generality, we assume $\bar{\gamma}(t) \in \bar{M}_\delta^+$ for all $t \in [-T, T]$. In this case, let $\bar{x}^- = \bar{\gamma}(-T)$ and $\bar{x}^+ = \Pi_{\ell=1}^k \mathbf{k}_\ell^- \mathbf{k}_\ell^+ \bar{\gamma}(T)$. By assumption, these two points can be connected by a curve ζ along which the action $A_L(\zeta) \leq A - kd$ as it follows from Proposition 6.1 that $A_L(\bar{\gamma}|_{[-T, T]}) \leq A$. Since k can be arbitrarily large, it implies the existence of a curve along which the action of L approaches to minus infinity, it also contradicts Proposition 6.1.

Let us begin with any $\bar{\gamma} \in \mathcal{G}(L)$. Given any small $\epsilon_i > 0$, by finitely many times of such surgery, we obtain a curve $\bar{\gamma}_i : \mathbb{R} \rightarrow \bar{M}$ with following properties:

1, for each small $\epsilon_i > 0$, $\bar{\gamma}(-T) \in \bar{M}_\delta^-$ and $\bar{\gamma}(T) \in \bar{M}_\delta^+$

$$A_L(\bar{\gamma}_i|_{[-T, T]}) \leq \inf_{\substack{T' \in \mathbb{R}_+ \\ \mathbf{k}^- \bar{\gamma}(-T) \in \bar{M}_\delta^- \\ \mathbf{k}^+ \bar{\gamma}(T) \in \bar{M}_\delta^+}} h_L^{T'}(\mathbf{k}^- \bar{\gamma}_i(-T), \mathbf{k}^+ \bar{\gamma}_i(T)) + \epsilon_i.$$

2, $\bar{\gamma}_i$ is smooth everywhere except for two points which fall beyond the region $\{x \in M : |x_1| \leq \Theta_i\}$, and $\Theta_i \rightarrow \infty$ as $\epsilon \rightarrow 0$.

Let $T_i > 0$ such that $\bar{\gamma}_{i1}(\pm T_i) = \pm \Theta_i$. Because of Lemma 6.2, we see that $T_i \rightarrow \infty$ as $\Theta_i \rightarrow \infty$. In virtue of the argument before, for any large $T \exists i_0 > 0$ such that the set $\{\bar{\gamma}_i|_{[-T, T]} : i \geq i_0\}$ is pre-compact in $C^1([-T, T], \bar{M})$. Let $T \rightarrow \infty$. By diagonal extraction argument, there is a subsequence of $\{\bar{\gamma}_i\}$ which converges C^1 -uniformly on each compact set to a C^1 -curve $\bar{\gamma} : \mathbb{R} \rightarrow \bar{M}$. Obviously, $\bar{\gamma} \in \mathcal{C}(L)$. \square

Theorem 6.3. *The map $L \rightarrow \mathcal{C}(L)$ is upper semi-continuous.*

Proof. Let $\bar{\gamma}_i \in \mathcal{C}(L_i)$, $L_i \rightarrow L$. If $\{\bar{\gamma}_i\}$ which converges C^1 -uniformly on each compact set to a C^1 -curve $\bar{\gamma}$, it is obvious that $\bar{\gamma} \in \mathcal{C}(L)$. \square

It is an immediate consequence of Definition 6.2 that

Proposition 6.2. *If the space-step Lagrangian L is actually periodic in x_1 , then a curve $\bar{\gamma} \in \mathcal{C}(L)$ if and only if its projection $\gamma = \pi\bar{\gamma} : \mathbb{R} \rightarrow M$ is semi-static.*

6.2. Local connecting orbits of type- c . An orbit $d\gamma$ (A curve γ) is said connecting one Aubry set $\tilde{\mathcal{A}}(c)$ to another one $\tilde{\mathcal{A}}(c')$ if the α -limit set of the orbit $d\gamma$ is contained in $\tilde{\mathcal{A}}(c)$ and the ω -limit set of the orbit $d\gamma$ is contained in $\tilde{\mathcal{A}}(c')$. It is called local connecting orbit if these two Aubry sets are close to each other. In contrast, it is called global connecting orbit when the two cohomology classes are far away from each other. In this subsection, we show how to construct local connecting orbits of type- c , which is used to handle the problem of incomplete intersection.

For this purpose, we use a new version of c -equivalence introduced in our previous work [LC]. The concept of c -equivalence was introduced in [Ma2] for the first time. The modified version is defined not on the whole M , but on a non-degenerate embedded $(n-1)$ -dimensional torus. We call Σ_c non-degenerately embedded $(n-1)$ -dimensional torus by assuming a smooth injection $\varphi : \mathbb{T}^{n-1} \rightarrow \mathbb{T}^n$ such that Σ_c is the image of φ , and the induced map $\varphi_* : H_1(\mathbb{T}^{n-1}, \mathbb{Z}) \rightarrow H_1(\mathbb{T}^n, \mathbb{Z})$ is an injection.

Let $\mathfrak{C} \subset H^1(\mathbb{T}^n, \mathbb{R})$ be a connected set where we are going to define c -equivalence. For each cohomology class $c \in \mathfrak{C}$, we assume that there exists a non-degenerate embedded $(n-1)$ -dimensional torus $\Sigma_c \subset \mathbb{T}^n$ such that each c -semi static curve γ transversally intersects Σ_c . Let

$$\mathbb{V}_c = \bigcap_U \{i_{U*} H_1(U, \mathbb{R}) : U \text{ is a neighborhood of } \mathcal{N}(c) \cap \Sigma_c\},$$

here $i_U : U \rightarrow M$ denotes inclusion map. \mathbb{V}_c^\perp is defined to be the annihilator of \mathbb{V}_c , i.e. if $c' \in H^1(\mathbb{T}^n, \mathbb{R})$, then $c' \in \mathbb{V}_c^\perp$ if and only if $\langle c', h \rangle = 0$ for all $h \in \mathbb{V}_c$. Clearly,

$$\mathbb{V}_c^\perp = \bigcup_U \{\ker i_U^* : U \text{ is a neighborhood of } \mathcal{N}(c) \cap \Sigma_c\}.$$

Note that there exists a neighborhood U of $\mathcal{N}(c) \cap \Sigma_c$ such that $\mathbb{V}_c = i_{U*}H_1(U, \mathbb{R})$ and $\mathbb{V}_c^\perp = \ker i_U^*$ (see [Ma2]).

We say that $c, c' \in H^1(M, \mathbb{R})$ are c -equivalent if there exists a continuous curve $\Gamma: [0, 1] \rightarrow \mathfrak{C}$ such that $\Gamma(0) = c$, $\Gamma(1) = c'$, $\alpha(\Gamma(s))$ keeps constant for all $s \in [0, 1]$, and for each $s_0 \in [0, 1]$ there exists $\delta > 0$ such that $\Gamma(s) - \Gamma(s_0) \in \mathbb{V}_{\Gamma(s_0)}^\perp$ whenever $s \in [0, 1]$ and $|s - s_0| < \delta$.

Let $\{e_i\}_{1 \leq i \leq n-1}$ be the standard basis of $H_1(\mathbb{T}^{n-1}, \mathbb{Z})$, $g_{i+1} = \varphi_*(e_i) \in H_1(\mathbb{T}^n, \mathbb{Z})$. Obviously, g_i is an n -dimensional integer vector for $i = 2, 3, \dots, n$. Because φ is injection, there exists $g_1 \in \mathbb{Z}^n$ such that the $n \times n$ matrix $G = (g_1, g_2, \dots, g_n)$ is uni-module, i.e. $\det G = \pm 1$. Using new coordinates $x \rightarrow G^{-1}x$, the Lagrangian $\tilde{L}(\dot{x}, x) = L(G\dot{x}, Gx)$ is also 2π -periodic in x . In new coordinates, let $\bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} = \{x_1 \in \mathbb{R}, (x_2, \dots, x_n) \in \mathbb{T}^{n-1}\}$ be a covering of \mathbb{T}^n , $\pi: \bar{M} \rightarrow M = \mathbb{T}^n$. The lift of Σ_c , $\pi^{-1}(\Sigma_c)$ has infinitely many compact components $\{\Sigma_c^i\}_{i \in \mathbb{Z}}$. If φ is linear, $\Sigma_c^i = \{x_1 = 2i\pi\}$.

Theorem 6.4. *Assume the cohomology class c^* is c -equivalent to the class c' through the path $\Gamma: [0, 1] \rightarrow H^1(\mathbb{T}^n, \mathbb{R})$. For each $s \in [0, 1]$, the following are assumed:*

1, *there exists a coordinate systems $G_s^{-1}x$ where the first component of rotation vector is positive $\omega_1(\mu_{\Gamma(s)})$ for each ergodic $\Gamma(s)$ -minimal measure $\mu_{\Gamma(s)}$;*

2, *for the covering space $\bar{M}_s = \mathbb{R} \times \mathbb{T}^{n-1}$ in this coordinate system the lift of non-degenerately embedded codimension-one torus $\Sigma_{\Gamma(s)}$ has infinitely many connected components, each of which is still a non-degenerately embedded codimension-one torus.*

Then there exist some classes $c^ = c_0, c_1, \dots, c_k = c'$ on this path, closed 1-forms η_i and $\bar{\mu}_i$ on M with $[\eta_i] = c_i$ and $[\bar{\mu}_i] = c_{i+1} - c_i$, and smooth functions ϱ_i on \bar{M} for $i = 0, 1, \dots, k-1$, such that the pseudo connecting curve set $\mathcal{C}(L_i)$ for the space-step Lagrangian*

$$L_i = L - \eta_i - \varrho_i \bar{\mu}_i$$

possesses the properties:

(i), *each curve $\bar{\gamma} \in \mathcal{C}(L_i)$ determines an orbit $(\gamma, \dot{\gamma})$ of ϕ_L^t ;*

(ii), *such orbit $(\gamma, \dot{\gamma})$ connects $\tilde{\mathcal{A}}(c_i)$ to $\tilde{\mathcal{A}}(c_{i+1})$, i.e., the α -limit set $\alpha(d\gamma) \subseteq \tilde{\mathcal{A}}(c_i)$ and ω -limit set $\omega(d\gamma) \subseteq \tilde{\mathcal{A}}(c_{i+1})$.*

Proof. By the definition of c -equivalence, there exists a path $\Gamma: [0, 1] \rightarrow H^1(M, \mathbb{R})$ with $\Gamma(0) = c^*$, $\Gamma(1) = c'$ such that for each $c = \Gamma(s)$ ($s \in [0, 1]$) on the path, there exists $\epsilon > 0$ such that $\Gamma(s') - c \in \mathbb{V}_{\Gamma(s)}^\perp$ whenever $s' \in [0, 1]$ and $|s - s'| < \epsilon$. Thus, there exist a non-degenerately embedded $(n-1)$ -dimensional torus Σ_c , a closed form $\bar{\mu}_c$ and a neighborhood U of $\mathcal{N}(c) \cap \Sigma_c$ such that $[\bar{\mu}_c] = \Gamma(s') - c$ and $\text{supp } \bar{\mu}_c \cap U = \emptyset$.

In the new coordinates $x \rightarrow G_c^{-1}x$ on the torus as before, the codimension one hypersurface Σ_c^0 separates \bar{M} into two parts, the upper part \bar{M}^+ and the lower part \bar{M}^- . Let $\Sigma_c^0 + \delta$ denotes the δ -neighborhood of Σ_c^0 in \bar{M} , we introduce a smooth function $\varrho \in C^r(\bar{M}, [0, 1])$ such that $\varrho = 0$ if $x \in \bar{M}^- \setminus (\Sigma_c^0 + \delta)$, $\varrho = 1$ if $x \in \bar{M}^+ \setminus (\Sigma_c^0 + \delta)$. Let η and $\bar{\mu}$ are closed 1-forms on M such that $[\eta] = c$ and $[\eta + \bar{\mu}] = c'$. These forms have natural lift on \bar{M} , denoted by the same symbol.

Clearly, a sufficiently small $\delta > 0$ can be chosen so that

$$(\Sigma_c^0 + \delta) \cap (\mathcal{C}(L + \eta) + 2\delta) \subset U,$$

It follows from the upper semi-continuity of $\mathcal{C}(L)$ w.r.t. L , we find

$$(6.3) \quad (\Sigma_c^0 + \delta) \cap (\mathcal{C}(L + \eta + \varrho\bar{\mu}) + \delta) \subset U,$$

if $\varrho\bar{\mu}$ is C^0 -sufficiently small. As $\varrho\bar{\mu}$ is carefully chosen so that its support is disjoint from U , each curve $\bar{\gamma} \in \mathcal{C}(L + \eta + \varrho\bar{\mu})$ is clearly a solution of the Euler-Lagrange equation determined by L , the term $\varrho\bar{\mu}$ has no contribution to the Euler-Lagrange equation along $\bar{\gamma}$. In other words, each curve in $\mathcal{C}(L + \eta + \varrho\bar{\mu})$ generates an orbit $d\gamma$ of $\phi_L^t: \mathbb{R} \rightarrow TM$.

The definition of \mathcal{C} tells us that for each curve $\bar{\gamma} \in \mathcal{C}$, $\gamma|_{(-\infty, t_0]}$ is backward semi-static once $\bar{\gamma}|_{(-\infty, t_0]}$ falls entirely into $\bar{M}^- \setminus (\Sigma_c^0 + \delta)$, $\gamma|_{[t_1, \infty)}$ is forward semi-static once $\bar{\gamma}|_{[t_1, \infty)}$ falls entirely into $\bar{M}^+ \setminus (\Sigma_c^0 + \delta)$. Therefore, $(\gamma(t), \dot{\gamma}(t)) \rightarrow \tilde{\mathcal{A}}(\Gamma(s))$ as $t \rightarrow -\infty$ and $(\gamma(t), \dot{\gamma}(t)) \rightarrow \tilde{\mathcal{A}}(\Gamma(s'))$ as $t \rightarrow \infty$.

Because of the compactness of $[0, 1]$, we can find finite set of numbers $s_0, \dots, s_k \in [0, 1]$ such that above argument applies if s and s' are replaced by s_i and s_{i+1} respectively. Set $c_i = \Gamma(s_i)$. \square

Corollary 6.1. *Let $c_i, \eta_i, \bar{\mu}_i$ and ϱ_i be evaluated as in Theorem 6.4. Let U_i be an open neighborhood of $\mathcal{N}(c_i) \cap \Sigma_{c_i}^0$ such that $U_i \cap \text{supp } \bar{\mu}_i = \emptyset$. Then, there exist big $K_i > 0$, $T_i > 0$ and small $\delta > 0$ such that for each $\bar{m}, \bar{m}' \in \bar{M}$, with $-K_i \leq \bar{m}_1 \leq -K_i + 2\pi$, $K_i - 2\pi \leq \bar{m}'_1 \leq K_i$, the quantity $h_{\eta_i, \mu_i}^T(\bar{m}, \bar{m}')$ reaches its minimum at some $T < T_i$ and the corresponding minimizer $\bar{\gamma}_i(t, \bar{m}, \bar{m}')$ satisfies the condition*

$$(6.4) \quad \text{Image}(\bar{\gamma}_i) \cap (\Sigma_{c_i}^0 + \delta) \subset U_i.$$

As the orbits obtained here is by using c -equivalence, we call them local connecting orbits of type- c . This type of connecting orbits are found in the annulus of incomplete intersection and plays key role in establishing transition chain crossing strong double resonance.

There is some flexibility to choose the coordinate system and the non-degenerately embedded codimension one torus. Let $\pi_s: \bar{M}_s \rightarrow M = \mathbb{T}^n$ be a covering space such that $\bar{M}_s = \mathbb{R} \times \mathbb{T}^{n-1}$ in the coordinate system $G_s^{-1}x$.

Definition 6.3. *For $s \in [0, 1]$, the non-degenerately embedded codimension one torus Σ_s is called admissible for the coordinate system $G_s^{-1}x$ if the lift of Σ_s to the covering space \bar{M}_s consists of infinitely many connected and compact components and the first component of the rotation vector is positive $\omega_1(\mu_{\Gamma(s)})$ for each ergodic $\Gamma(s)$ -minimal measure.*

Let us describe how the equivalence relation is established between two classes near strong double resonance. Let $\Gamma \subset \mathbb{A} \subset \alpha^{-1}(E)$ be a curve skirting around the flat \mathbb{F}_0 , along which the α -function keeps constant and the third coordinate c_3 keeps constant as well. For each $c \in \Gamma$, one can find certain coordinate system and finitely many intervals $I_{c,i}$ such that each c -semi static curve passes through the section $\Sigma_c = \{x_1 = 0\}$ transversally and

$$\mathcal{N}(c) \cap \Sigma_c \subset \{(x_2, x_3) : x_2 \in \cup I_{c,i}, x_3 \in \mathbb{T}\}.$$

Clearly, some open set $U \supset \mathcal{N}(c) \cap \Sigma_c$ such that $V_c = i_{U*}H_1(U, \mathbb{R}) = \text{span}\{(0, 0, 1)\}$, from which one obtains that $V_c^\perp = \text{span}\{(1, 0, 0), (0, 1, 0)\}$. For each class $c' \in \Gamma$ very close to c , one has $c' - c = (\Delta c_1, \Delta c_2, 0) \in V_c^\perp$, thus, there exists a closed 1-form $\bar{\mu}$ such that $[\bar{\mu}] = c' - c$ and

$$\text{supp} \bar{\mu} \cap \mathcal{N}(c) \cap \Sigma_c = \emptyset.$$

Thus, any two classes along the curve Γ are equivalent.

6.3. Local connecting orbits of type- h . Another type of local connecting orbits look like heteroclinic orbits. Therefore, we call them local connecting orbits of type- h .

It is used to handle a typical case when an Aubry set falls in a neighborhood of lower dimensional torus N such that $H_1(M, N, \mathbb{Z}) \neq 0$. Equivalence relation seems not exist among those classes if the Aubry sets fall into N . However, each of these Aubry sets has homoclinic orbit, it may lead to the existence of heteroclinic orbits. Towards this goal, let us work in suitable finite covering manifold $\tilde{\pi}: \tilde{M} \rightarrow M$. In this covering space, these homoclinic orbits turn out to be semi-static orbits. We assume that the Aubry set $\mathcal{A}(c, \tilde{M})$ consists of finitely many classes $\mathcal{A}(c) = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$ ($k > 1$), \tilde{M} is chosen so that the lift of N , $\tilde{N} = N_1 \cup \dots \cup N_k$ with $k > 1$, $\tilde{\pi}N_i = N$ and $N_i \cap N_j = \emptyset$ provided $i \neq j$. In the following, we denote by N_i an open domain such that each N_i contains one Aubry class $N_i \supset \mathcal{A}_i$.

If a Aubry set contains finitely many static classes only, denoted by $\tilde{\mathcal{A}}_i$ ($i = 1, 2, \dots, k$), then these classes are transitive in the following sense: by rearranging the subscripts, there exist k semi-static curves $\gamma_{i,i+1} \pmod{k}$ such that $\omega(d\gamma_{i,i+1}) \subseteq \tilde{\mathcal{A}}_{i+1}$ and $\alpha(d\gamma_{i,i+1}) \subseteq \tilde{\mathcal{A}}_i$ [CP]. It does not exclude the case that some semi-static curve $\gamma_{i,j}$ exists such that $j \neq i + 1 \pmod{k}$, $\alpha(d\gamma_{i,j}) \subseteq \tilde{\mathcal{A}}_i$ and $\omega(d\gamma_{i,j}) \subseteq \tilde{\mathcal{A}}_j$. We say that $\tilde{\mathcal{A}}_i$ is connected to $\tilde{\mathcal{A}}_j$ through $\tilde{\mathcal{A}}_{i'}$ with $i' = i + 1, i + 2, \dots, j - 1$ if there exist semi-static curves $\gamma_{i',i'+1}$ such that $\omega(d\gamma_{i',i'+1}) \subseteq \tilde{\mathcal{A}}_{i'+1}$ and $\alpha(d\gamma_{i',i'+1}) \subseteq \tilde{\mathcal{A}}_{i'}$.

The Aubry set \mathcal{A}_i is said to be directly connected to the Aubry set \mathcal{A}_j if a semi-static curve $\gamma: \mathbb{R} \rightarrow M$ exists such that $\omega(d\gamma) \subseteq \tilde{\mathcal{A}}_j$ and $\alpha(d\gamma) \subseteq \tilde{\mathcal{A}}_i$. That \mathcal{A}_i is directly connected to \mathcal{A}_j does not imply that \mathcal{A}_j is directly connected to \mathcal{A}_i .

Pick up two points $x_i \in \mathcal{A}_i$, $x_j \in \mathcal{A}_j$, we consider the quantity

$$h_c^T(x_i, x_j) = \inf_{\substack{\gamma(-T)=x_i \\ \gamma(T)=x_j}} \int_{-T}^T L_c(d\gamma(t))dt + 2T\alpha(c).$$

By standard notation,

$$h_c^\infty(x_i, x_j) = \liminf_{T \rightarrow \infty} h_c^T(x_i, x_j).$$

Let $\gamma^T: [-T, T] \rightarrow M$ be the minimizer realizing the quantity $h_c^T(x_i, x_j)$. Let $[t_{i,T}, t_{j,T}]$ be the sub-interval of $[-T, T]$ such that $\gamma^T(t) \notin N_i \cup N_j$ for $t \in (t_{i,T}, t_{j,T})$ but $\gamma^T(t_{i,T}) \in \tilde{N}_i$ and $\gamma^T(t_{j,T}) \in \tilde{N}_j$. In the case that \mathcal{A}_i is directly connected only to \mathcal{A}_j , $t_{j,T} - t_{i,T}$ is upper bounded uniformly for $T > 0$. Some sequence of time t_T and a positive number $\Delta > 0$ such that $[t_T - \Delta, t_T + \Delta] \subset (t_{i,T}, t_{j,T})$ for sufficiently large T . The set of curves $\{\gamma^T(t - t_T)|_{[-\Delta, \Delta]}\}$ is compact in C^1 -topology. Let $\gamma|_{[-\Delta, \Delta]}$ be the accumulation point which can be uniquely extended to whole line $\gamma: \mathbb{R} \rightarrow M$. Clearly, $\alpha(d\gamma) \subset \tilde{\mathcal{A}}_i$ and $\omega(d\gamma) \subset \tilde{\mathcal{A}}_j$. If \mathcal{A}_i is directly connected also to other \mathcal{A}_k , one can also obtain such a sequence of curves by introducing small perturbation so

that \mathcal{A}_i is directly connected only to \mathcal{A}_j and the support of the perturbation does not touch the semi-static curves connecting \mathcal{A}_i to \mathcal{A}_j .

Given a semi-static curve one can choose an $(n-1)$ -dimensional disk Σ such that they intersects each other transversally. This disk also intersects semi-static curves nearby. A semi-static curve is said *disconnected* to other semi-static curves if the intersection point is disconnected to the intersection points of all other semi-static curves.

Theorem 6.5. (Connecting Lemma) *Assume that the Aubry set contains finitely many classes $\mathcal{A}(c) = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_k$, there exist open domains $N_1 \dots N_k$ such that $\mathcal{A}_i \subset N_i$ for each $1 \leq i \leq k$ and $N_i \cap N_j = \emptyset$ provided $i \neq j$. If each semi-static curves connecting different Aubry sets is disconnected to all other semi-static curves, then there exists some orbit $d\gamma'$ of ϕ_L^t connecting $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$ provided $\alpha(c) = \alpha(c')$, the class c' is sufficiently close to the class c , $\mathcal{A}(c') \subset \cup_{i=1}^k N_i$, N_i and N_j exist such that $\mathcal{A}(c') \cap N_i \neq \emptyset$ and $\mathcal{A}(c') \cap N_j \neq \emptyset$.*

Proof. In autonomous case, $\tilde{\mathcal{A}}(c)$ can be connected to $\tilde{\mathcal{A}}(c')$ only if $\alpha(c) = \alpha(c')$. If both c and c' are the minimal points of the α -function, then $\tilde{\mathcal{A}}(c) \cap \tilde{\mathcal{A}}(c') \neq \emptyset$ (see [Ms]), it is trivial to connect an Aubry set to itself. Thus we only need to work on the energy surface $H^{-1}(E)$ with $E > \min \alpha$, the minimum of the α -function. In this case, we obtain from [Lx] that

Proposition 6.3. *Let $L : \mathbb{T}^n \rightarrow \mathbb{R}$ be an autonomous Lagrangian of Tonelli type, the class c be not the minimal point of the α -function, and Ω_c be the flat of the β -function such that*

$$\omega \in \Omega_c \Rightarrow \alpha(c) + \beta(\omega) = \langle c, \omega \rangle.$$

Then, there exists a coordinate system such that each rotation vector in this flat has positive first component $\omega_1 > 0$.

The existence of such connecting orbits is derived from the upper-semi continuity of so called pseudo-connecting orbit set introduced in Definition 6.2. For the definition of this set in autonomous case, we need to work in certain covering space $\pi : \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1}$ if $\omega_1(\mu_c) > 0$ holds for each ergodic minimal measure μ_c . By Proposition 6.3, it is possible if we choose suitable coordinate system. Let $\bar{\gamma}$ denote the lift of the curve $\gamma : \mathbb{R} \rightarrow M$, $\bar{\gamma}_1$ denote the first coordinate.

Let $\Sigma_0 = \{x : x_1 = 0\}$ be a codimension one hyperplane separating \bar{M} into two parts, the upper part \bar{M}^+ connecting to $\{x_1 = \infty\}$ and the lower part \bar{M}^- connecting to $\{x_1 = -\infty\}$. Let $\Sigma_0 + \delta$ denote the δ -neighborhood of Σ_0 in \bar{M} , we introduce a smooth function $\rho \in C^r(\bar{M}, [0, 1])$ such that $\rho = 0$ if $x \in \bar{M}^- \setminus (\Sigma_0 + \delta)$, $\rho = 1$ if $x \in \bar{M}^+ \setminus (\Sigma_0 + \delta)$. Let η and $\bar{\mu}$ be closed 1-forms on M such that $[\eta] = c$ and $[\eta + \bar{\mu}] = c'$. They have natural lift on \bar{M} . Let $\mu = \rho\bar{\mu}$. We carefully choose smooth function $\psi = \psi(x, \dot{x})$ such that $\psi = 0$ as $x_1 \in (-\infty, -1) \cup (1, \infty)$ (the construction is demonstrated later) and let

$$L_{\eta, \mu, \psi} = L - \eta - \mu - \psi.$$

Let \bar{m}, \bar{m}' be two points in \bar{M} , we define

$$h_{\eta, \mu, \psi}^T(\bar{m}, \bar{m}') = \inf_{\substack{\gamma(-T) = \bar{m} \\ \gamma(T) = \bar{m}'}} \int_{-T}^T L_{\eta, \mu, \psi}(d\gamma(t)) dt.$$

For small μ and ψ , the Lagrangian $L_{\eta,\mu,\psi}$ satisfies the conditions required for space-step Lagrangian. In the following we shall use the notation $\mathcal{C}_{\eta,\mu,\psi} = \mathcal{C}(L_{\eta,\mu,\psi})$ to denote relevant set of pseudo-connecting curves.

According to Proposition 6.3, it is reasonable to assume $\omega_1(\mu_c) > 0$ for each ergodic minimal measure by choosing suitable coordinate system. We work in the covering space $\pi : \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} \rightarrow M$.

Let us recall a graph property. Given two Aubry classes $\tilde{\mathcal{A}}_i$ and $\tilde{\mathcal{A}}_j$, let $\tilde{\mathcal{N}}_{ij}$ be the set of all semi-static orbits whose α -limit set is in $\tilde{\mathcal{A}}_i$ and the ω -limit set is in $\tilde{\mathcal{A}}_j$. Let $\mathcal{N}_{ij} = \pi_x \tilde{\mathcal{N}}_{ij}$, where π_x denotes the standard projection $TM \rightarrow M$. Then, the inverse of π_x , restricted on \mathcal{N}_{ij} , is Lipschitz. The proof is the same as it for the graph property of Aubry set.

By assumption, there exists a semi-static orbit $d\zeta_{ij}$ connecting $\tilde{\mathcal{A}}_i$ to $\tilde{\mathcal{A}}_j$ if $\tilde{\mathcal{A}}_i$ is directly connected to $\tilde{\mathcal{A}}_j$. By permutating the subscripts, we can assume that $\tilde{\mathcal{A}}_i$ is directly connected at least to $\tilde{\mathcal{A}}_{j+1}$. Let $\tilde{\zeta}_{ij}$ denote a lift of ζ_{ij} to \bar{M} . Pick up a point $x_0 = (x_{0,1}, \dots, x_{0,n}) = \tilde{\zeta}_{ij}(t_0) \notin \overline{UN_i}$ with $x_{0,1} = 0$ and denote by $v_0 = \dot{\tilde{\zeta}}_{ij}(t_0)$ the velocity of $\tilde{\zeta}_{ij}$ at x_0 . Obviously, $v_0 \neq 0$. Let ϱ' be a smooth function in s such that $\varrho' = 0$ for $s \leq 0$, $\varrho' = 1$ for $s > \delta$ and $\varrho' > 0$ for $s \in (0, \delta)$, where $\delta > 0$ is suitably small. Let $\varrho_{ij}(x) = \varrho'(\langle x - x_0, v_0 \rangle)$, then $\langle \partial \varrho_{ij}(x), v \rangle = \langle v_0, v \rangle \varrho'(s)$ provided $\langle x - x_0, v_0 \rangle = s$.

We choose an $(n-1)$ -dimensional plane $\Pi_{ij,s} = \{x : \langle x - x_0, v_0 \rangle = s\}$. Since the set of semi-static curves is totally disconnected, we can choose, for each $s \in [0, \delta]$, two suitably small $(n-1)$ -dimensional topological disks $D'_{ij,s}, D_{ij,s}$ located in $\Pi_{ij,s}$ and small $\delta_1 > 0$ such that $D'_{ij,s} \cap (\cup N_j) = \emptyset$, $D'_{ij,s} \supset D_{ij,s} + \delta_1$, ζ_{ij} passes through the disk $D_{ij,s}$ and no semi-static curve in \mathcal{N}_{ij} passes through $D'_{ij,s} \setminus D_{ij,s}$. These disks can be chosen so that the Hausdorff distance $d_H(D_{ij,s}, D_{ij,s'}) \rightarrow 0$ and $d_H(D'_{ij,s}, D'_{ij,s'}) \rightarrow 0$ as $s' \rightarrow s$. Let $D'_{ij} = \cup_{s \in [0, \delta]} D'_{ij,s}$, $D_{ij} = \cup_{s \in [0, \delta]} D_{ij,s}$. We choose a smooth non-negative function $w_{ij} : \bar{M} \rightarrow \mathbb{R}$ such that $\text{supp} w_{ij} \cap \{x : 0 \leq \langle x - x_0, v_0 \rangle \leq \delta\} = D'_{ij}$ and $w_{ij} \equiv \lambda$ if $x \in D_{ij}$. As it will be seen later, we do not care about how w_{ij} is defined on those $(n-1)$ -dimensional plane $\Pi_{ij,s}$ with $s \notin [0, \delta]$.

For different $(i, j) \neq (i', j')$, it is possible that $D_{ij} \cap \mathcal{N}_{i',j'} \neq \emptyset$. But it does not make trouble, as $\tilde{\mathcal{N}}_{ij} \cap \tilde{\mathcal{N}}_{i',j'} = \emptyset$. Let S_{ij} be the graph of a Lipschitz map $x \rightarrow \dot{x}$ such that $\tilde{\mathcal{N}}_{ij}$ is located within. Thus, we choose a smooth function $v_{ij} : TM \rightarrow [0, 1]$ such that $v_{ij} \equiv 1$ when $(x, \dot{x}) \in (S_{ij} + \delta_2) \cap TD'_{ij}$ and $v_{ij} \equiv 0$ when $(x, \dot{x}) \notin S_{ij} + \delta_3$, where $\delta_3 > \delta_2 > 0$ are small numbers. As there are finitely many Aubry classes, we have $\text{supp} v_{ij} \cap \text{supp} v_{i',j'} = \emptyset$ if $(i, j) \neq (i', j')$.

Let us consider what curves contained in the set $\mathcal{C}_{\eta,0,\psi}$ defined for the Lagrangian $L_{\eta,0,\psi}$ by assuming

$$\psi = \sum v_{ij} w_{ij} \langle \partial \varrho_{ij}, \dot{x} \rangle.$$

By the upper semi-continuity of $L \rightarrow \mathcal{C}(L)$, each curve $\bar{\gamma} \in \mathcal{C}_{\eta,0,\psi}$ stays in a small neighborhood of certain curve $\bar{\gamma}_{ij}$, which is a lift of certain semi-static curve γ_{ij} , provided $\|w_{ij}\|_{C^2}$ is suitably small. Let us pick up a semi-static curve $\zeta_{ij} : \mathbb{R} \rightarrow \mathbb{T}^n$ for L_η and consider one component of its lift $\tilde{\zeta}_{ij} : \mathbb{R} \rightarrow \bar{M}$ which passes through D_{ij} . By the notation, the orbit $(\zeta_{ij}, \dot{\zeta}_{ij})$ connects $\tilde{\mathcal{A}}_i$ to $\tilde{\mathcal{A}}_j$. Since the 1-form $w_{ij} \langle \partial \varrho_{ij}, dx \rangle$

is closed in D_{ij} , $\langle \partial \varrho_{ij}, \dot{x} \rangle = \dot{\varrho}'(s) \langle \dot{x}, v_0 \rangle = 0$ on each $\Pi_{ij,s}$ with $s \notin [0, \delta]$ this term has no contribution to the Euler-Lagrange equation along this semi-static curve, namely, this curve solves the Euler-Lagrange equation determined by $L_{\eta,0,\psi}$ also. However, the action of $L_{\eta,0,\psi}$ along this curve is smaller than the action of $L_{\eta,0,\psi}$ since ψ is non-negative and takes positive value along a piece of the curve.

The lift of the curve ζ_{ij} to \bar{M} contains infinitely many curves $\{\bar{\zeta}_{ij} + (k, 0, \dots, 0) : k \in \mathbb{Z}\}$ where $\pi \bar{\zeta}_{ij} = \zeta_{ij}$. Each of these curves solves the Euler-Lagrange equation determined by $L_{\eta,0,\psi}$, but except $\bar{\zeta}_{ij}$, all other curves do not belong to $\tilde{\mathcal{C}}_{\eta,0,\psi}$ since they do not pass through πD_{ij} , the action along these curves is bigger than the action along $\bar{\zeta}_{ij}$. Since no semi-static curve in \mathcal{N}_{ij} passes through $(D'_{i,j} \setminus D_{ij})$, each curve in $\mathcal{C}_{\eta,0,\psi}$ solves the Euler-Lagrange equation determined by $L_{\eta,0,\psi}$. Thus, one has $\bar{\zeta}_{ij} \in \mathcal{C}_{\eta,0,\psi}$. As $L_{\eta,0,\psi} = L_{\eta}$ when they are restricted to $T(\cup N_i)$, every curve in the lift of a static curve for L_{η} is clearly in $\tilde{\mathcal{C}}_{\eta,0,\psi}$.

In the cylinder \bar{M} there exist two sections Σ^+ and Σ^- satisfying the following conditions:

1, both are the deformation of $\{x : x_1 = 0\}$, they divide \bar{M} into three parts, \bar{M}^+ , \bar{M}^- and \bar{M}_0 bounded by $\Sigma_+ \cup \Sigma_-$. \bar{M}^+ is homeomorphic $(0, \infty) \times \mathbb{T}^{n-1}$, \bar{M}^- is homeomorphic $(-\infty, 0) \times \mathbb{T}^{n-1}$ and \bar{M}_0 is homeomorphic to $(0, 1) \times \mathbb{T}^{n-1}$;

2, there exists $\delta_4 > 0$ such that $\cup D'_{ij} + \delta_4 \subset \bar{M}_0$;

3, for each $\bar{\zeta}_{ij} \in \mathcal{C}_{\eta,0,\psi}$, both $\text{Image} \bar{\zeta} \cap \bar{M}^+$ and $\text{Image} \bar{\zeta} \cap \bar{M}^-$ are connected, i.e. if one moves into \bar{M}_{\pm} along the curve as $t \rightarrow \pm\infty$ then it stays in \bar{M}_{\pm} forever.

Let U_{ij}^+ be the tube connecting D_{ij} to \bar{M}^+ , $U_{ij}^+ \cap D_{ij} = D_{ij,\delta}$, each $\bar{\zeta}_{ij} \in \mathcal{C}_{\eta,0,\psi}$ passes through U_{ij}^+ , does not touch the boundary of U_{ij}^+ before it moves forward into \bar{M}^+ . Similarly, we define the tube U_{ij}^- connecting D_{ij} to \bar{M}^- such that $U_{ij}^- \cap D_{ij} = D_{ji,0}$, each of those curve passes through U_{ij}^- , does not touch the boundary of U_{ij}^- before it returns back into \bar{M}^- .

Since there are finitely many Aubry classes only, by choosing suitably small D'_{ij} we can assume $D'_{ij} \cap D'_{i',j'} = \emptyset$ if $(i, j) \neq (i', j')$. A closed 1-form $\bar{\mu}$ clearly exists such that $[\bar{\mu}] = c' - c$ and $\text{supp} \bar{\mu} \cap (\cup D_{ij}) = \emptyset$. Let $\rho' : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\rho' = 0$ for $s \leq 0$, $\rho' = 1$ for $s > \delta$ and let U'_{ij} be an open set containing the closure of $U_{ij}^+ \cup D'_{ij} \cup U_{ij}^-$ and $U'_{ij} \cap U'_{i',j'} = \emptyset$ if $(i, j) \neq (i', j')$. We define a smooth function $\rho : \bar{M} \rightarrow [0, 1]$ such that $\rho(x) = \rho'(\langle x - x_0, v_0 \rangle)$ if $x \in D_{ij}$ where $x_0 = \bar{\zeta}_{ij}(t_0)$ and $v_0 = \dot{\bar{\zeta}}_{ij}(t_0)$, $\rho = 1$ if $x \in \bar{M}^+ \cup (\cup U_{ij}^+)$ and $\rho(x) = 0$ if $x \in \bar{M}^- \cup (\cup U_{ij}^-)$. By the construction of \bar{M}^{\pm} , U_{ij}^{\pm} and $D_{i,j}$, we see the existence of such function.

Let us now study the Lagrangian $L_{\eta,\mu,\psi}$ with $\mu = \rho \bar{\mu}$. By condition, $\mathcal{A}(c') \cap N_i \neq \emptyset$, $\mathcal{A}(c') \cap N_j \neq \emptyset$ and $i \neq j$. Thus, there exist $x_i \in \mathcal{M}(c) \cap N_i$ and $x_j \in \mathcal{M}(c') \cap N_j$. Let \bar{x}_i and \bar{x}_j be two points in \bar{M} such that $\pi \bar{x}_i = x_i$ and $\pi \bar{x}_j = x_j$ and let $\bar{x}_{ik} = \bar{x}_i - k e_1$ and $\bar{x}_{jk} = \bar{x}_j + k e_1$ where $e_1 = (1, 0, \dots, 0)$. Let $\bar{\gamma}_k : [-T, T] \rightarrow \bar{M}$ be the minimizer of

$$\inf_{T' > 0} h_{\eta,\mu,\psi}^{T'}(\bar{x}_{ik}, \bar{x}_{jk}) = \int_{-T}^T L_{\eta,\mu,\psi}(d\bar{\gamma}_k(t)) dt + 2T\alpha(c),$$

and let $k \rightarrow \infty$, we obtain a sequence of $\{\bar{\gamma}_k\}$. Let $\bar{\gamma}: \mathbb{R} \rightarrow \bar{M}$ be the accumulation point of the sequence. Due to the upper semi-continuity of $\mathcal{C}_{\eta,\mu,\psi}$ with respect to (η, μ, ψ) , the curve $\bar{\gamma}$ must pass through $\cup D_{ij}$ if $|c' - c|$ is suitably small. Thus, along the curve $\bar{\gamma}$ the term $\rho\bar{\mu}$ does not contribute the Lagrange equation, namely, the curve determines an orbit of ϕ_L^t . Since this curve is in the set $\mathcal{C}_{\eta,\mu,\psi}$, therefore, it connects $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$. This completes the proof. \square

6.4. Locally minimal property. The orbit $d\gamma$ obtained in Theorem 6.5 is locally minimal in the sense we define in the following. It is crucial for the variational construction of global connecting orbits. The set of local minimal curve will be not empty if the Aubry set $\mathcal{A}(c)$ has some totally disconnected minimal homoclinic orbit and μ as well as ψ is carefully chosen for the modified Lagrangian.

Here is the definition for autonomous systems:

Definition 6.4. Let $N_1, \dots, N_k \subset M$ be open domains such that $\text{dist}(N_i, N_j) > 0$ ($k > 1$). We assume that $\mathcal{A}(c), \mathcal{A}(c') \subset \cup N_i$, $[\eta] = c$, $[\eta + \bar{\mu}] = c'$, $\alpha(c) = \alpha(c')$ and the first component of both c - and c' -minimal measures is positive $\omega_1(\mu_c) > 0$, $\omega_1(\mu_{c'}) > 0$. Let $\pi: \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} \rightarrow M$ be the covering space, denote by $\bar{\gamma}$ the lift of a curve $\gamma: \mathbb{R} \rightarrow M$. Then, $d\gamma: TM \rightarrow \mathbb{R}$ is called local minimal orbit of type- h that connects $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$ if

1, $d\gamma$ is an orbit of ϕ_L^t , $\alpha(d\gamma) \subset \tilde{\mathcal{A}}(c)$ and $\omega(d\gamma) \subset \tilde{\mathcal{A}}(c')$. Some neighborhoods of co-dimension one torus N_i, N_j exist such that $\alpha(d\gamma) \subset TN_i$ and $\omega(d\gamma) \subset TN_j$;

2, there exist two $(n-1)$ dimensional disks $V_i^-, V_j^+ \subset \bar{M}$ and positive numbers $T, d > 0$ such that $\pi V_i^- \subset N_i \setminus \mathcal{A}(c)$, $\pi V_j^+ \subset N_j \setminus \mathcal{A}(c')$, γ transversally passes πV_i^- and πV_j^+ at the time $-T$ and T respectively, and

$$(6.5) \quad h_c^\infty(x^-, \pi \bar{m}_0) + h_{\eta,\mu,\psi}^{T'}(\bar{m}_0, \bar{m}_1) + h_{c'}^\infty(\pi \bar{m}_1, x^+) \\ - \lim_{\substack{t_i^- \rightarrow \infty \\ t_i^+ \rightarrow \infty}} \int_{-t_i^-}^{t_i^+} L_{\eta,\mu,\psi}(d\gamma(t)) dt - (t_i^- + t_i^+) \alpha(c) > 0$$

holds for each $(\bar{m}_0, \bar{m}_1, T') \in \partial(V_i^- \times V_j^+ \times [T-d, T+d])$, $x^- \in N_i \cap \pi_x(\alpha(d\gamma))$ and $x^+ \in N_j \cap \pi_x(\omega(d\gamma))$. Where $t_i^- \rightarrow \infty$ and $t_i^+ \rightarrow \infty$ are the sequences such that $\gamma(-t_i^-) \rightarrow x^-$ and $\gamma(t_i^+) \rightarrow x^+$. Where $\pi_x: TM \rightarrow M$ is the standard projection.

In this definition, the term $h_c^\infty(x^-, \pi \bar{m}_0) + h_{\eta,\mu,\psi}^{T'}(\bar{m}_0, \bar{m}_1) + h_{c'}^\infty(\pi \bar{m}_1, x^+)$ measures the smallest action of $L_{\eta,\mu,\psi}$ along those curves $\{\chi\}$ which satisfy the following conditions:

- 1, each χ joining m_0 with m_1 with time $2T'$;
- 2, x^- is an accumulation point of χ as $t \rightarrow -\infty$ and x^+ is an accumulation point of χ as $t \rightarrow \infty$.

Remark. In the space of curves, a neighborhood of the curve γ consists of those curves that start from V^- and reach V^+ within a time between $2(T-d)$ and $2(T+d)$. The minimal curves connecting from V^- to V^+ with different time scale determine

orbits in different energy levels, that is why we consider the time scale $T' \in [T - d, T + d]$ as variable when we search for the local minimum.

Remark. This definition applies also to the case that there exists only one Aubry class staying in the small neighborhood of lower-dimensional torus. In that case, we can consider a suitable finite covering of the configuration manifold. In the finite covering configuration space, there are more than one Aubry class.

The following is for time-periodic systems

Definition 6.5. Let $N_1, \dots, N_k \subset M$ ($k > 1$) be open domains such that each of them keeps positive distance with any other one: $\text{dist}(N_i, N_j) > 0$. We assume that $\mathcal{A}_0(c), \mathcal{A}_0(c') \subset \cup N_i$, $[\eta] = c$, $[\eta + \bar{\mu}] = c'$. Then, $d\gamma: TM \rightarrow \mathbb{R}$ is called local minimal orbit of type- h that connects $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$ if

1, $d\gamma$ is an orbit of ϕ_L^t , the α -limit and the ω -limit sets of $d\gamma$ are contained in $\tilde{\mathcal{A}}(c)$ and $\tilde{\mathcal{A}}(c')$ respectively, $\alpha(d\gamma)|_{t=0} \subset TN_i$ and $\omega(d\gamma)|_{t=0} \subset TN_j$ with $i \neq j$;

2, there exist two open balls V_i^-, V_j^+ and two positive integers t^-, t^+ such that $\bar{V}_j^- \subset N_i \setminus \mathcal{M}_0(c)$, $\bar{V}_j^+ \subset N_j \setminus \mathcal{M}_0(c')$, $\gamma(-k^-) \in V_i^-$, $\gamma(k^+) \in V_j^+$ and

$$(6.6) \quad \begin{aligned} & h_c^\infty(x^-, m_0) + h_{\eta, \mu, \psi}^{k^-, k^+}(m_0, m_1) + h_{c'}^\infty(m_1, x^+) \\ & - \liminf_{\substack{k_i^- \rightarrow \infty \\ k_i^+ \rightarrow \infty}} \int_{-k_i^-}^{k_i^+} L_{\eta, \mu, \psi}(d\gamma(t), t) dt - k_i^- \alpha(c) - k_i^+ \alpha(c') \\ & > 0 \end{aligned}$$

holds for any $(m_0, m_1) \in \partial(V_i^- \times V_j^+)$, $x^- \in N_i \cap \pi_x(\alpha(d\gamma))|_{t=0}$, $x^+ \in N_j \cap \pi_x(\omega(d\gamma))|_{t=0}$ and k_i^-, k_i^+ are the sequences such that $\gamma(-k_i^-) \rightarrow x^-$ and $\gamma(k_i^+) \rightarrow x^+$. Where $\pi_x: TM \rightarrow M$ is the standard projection.

The set of curves starting from V^- and reaching V^+ with time $k^- + k^+$ constitutes a neighborhood of the curve γ in the space of curves. Once a curve $\tilde{\gamma}$ touches the boundary of this neighborhood, the action of $L_{\eta, \mu, \psi}$ along $\tilde{\gamma}$ will be larger than the action along γ . As V^-, V^+ and therefore $d > 0$ can be chosen arbitrarily small, it is reasonable to call it locally minimal.

7. VARIATIONAL CONSTRUCTION OF GLOBAL CONNECTING ORBITS

The task of this section is showing how to construct global connecting orbits by variational method, provided a generalized transition chain exists. In the next section, the main result (Theorem 1.1) is proved by showing the genericity of such transition chain.

7.1. Generalized Transition chain. The concept of transition chain was proposed by Arnold in his celebrated paper [Ar1] where it is formulated in geometric language. The generalized transition chain formulated in our previous work [CY1, CY2] is in variational version which need less information about the geometric structure.

Definition 7.1. (Autonomous Case) *Let c, c' be two cohomology classes in $H^1(M, \mathbb{R})$. We say that c is joined with c' by a generalized transition chain if a continuous curve $\Gamma: [0, 1] \rightarrow H^1(M, \mathbb{R})$ exists such that $\Gamma(0) = c$, $\Gamma(1) = c'$, $\alpha(\Gamma(s)) \equiv E > \min \alpha$ and the condition is satisfied: for each $s \in [0, 1]$ at least one of the following cases takes place:*

(H1), *the Aubry set is composed of finitely many classes only. There exist certain finite covering: $\tilde{\pi}: \tilde{M} \rightarrow M$, two open domains N_1, N_2 with $d(N_1, N_2) > 0$, an $(n-1)$ dimensional disk Σ_s and small number $\delta_s > 0$ such that*

i, *the Aubry set $\mathcal{A}(\Gamma(s)) \cap N_1 \neq \emptyset$ and $\mathcal{A}(\Gamma(s)) \cap N_2 \neq \emptyset$ and $\mathcal{A}(\Gamma(s')) \subset N_1 \cup N_2$ for each $|s' - s| < \delta_s$,*

ii, *$\tilde{\pi}\mathcal{N}(\Gamma(s), \tilde{M})|_{\Sigma_s} \setminus (N_1 \cup N_2)$ is non-empty and totally disconnected;*

(H2), *For each $s' \in (s - \delta_s, s + \delta_s)$, $\Gamma(s')$ is equivalent to $\Gamma(s)$, namely, some section Σ_s and some neighborhood of $\mathcal{N}(\Gamma(s)) \cap \Sigma_{\Gamma(s)}$, denoted by U , exist such that $\Gamma(s') - \Gamma(s) \in \ker i_U^*$. Each class $\Gamma(s')$ is associated with an admissible section $\Sigma_{s'}$ and an admissible coordinate system $G_{s'}^{-1}x$.*

Remark. Because of upper semi-continuity of Mañé set, it is possible that there exist some classes for which both cases take place.

In the case (H1), if the Aubry set contains only one Aubry class, one can take some finite covering $\tilde{\pi}: \tilde{M} \rightarrow M$ non trivial provided $H_1(M, \mathcal{A}, \mathbb{R}) \neq 0$. A typical case is that $\mathcal{A}(\Gamma(s))$ is homeomorphic to co-dimension one torus, one takes suitable finite covering space so that $\mathcal{A}(\Gamma(s), \tilde{M})$ contains exactly two connected components. If $\mathcal{A}(\Gamma(s))$ contains more than one class, we may take $\tilde{M} = M$.

By the study in the last section, the existence of generalized transition chain implies that one has a sequence of local connecting orbits. More precisely, there exists a sequence of locally minimal curve γ_i , a sequence of numbers s_i ($s = 0, 1, \dots, m$) such that $\alpha(d\gamma_i) \subset \mathcal{A}(\Gamma(s_i))$ and $\omega(d\gamma_i) \subset \mathcal{A}(\Gamma(s_{i+1}))$. Global connecting orbits are constructed shadowing these local connecting orbits.

One can also define generalized transition chain for time-periodic systems.

Definition 7.2. (Time-periodic Case) *Let c, c' be two classes in $H^1(M, \mathbb{R})$. We say that c is joined with c' by a generalized transition chain if a continuous curve $\Gamma: [0, 1] \rightarrow H^1(M, \mathbb{R})$ exists such that $\Gamma(0) = c$, $\Gamma(1) = c'$ and for each $s \in [0, 1]$ at least one of the following cases takes place:*

(H1), *the Aubry set is composed of finitely many classes only. There exist certain finite covering: $\tilde{\pi}: \tilde{M} \rightarrow M$, two open domains N_1, N_2 with $d(N_1, N_2) > 0$ and small number $\delta_s > 0$ such that*

i, *the Aubry set $\mathcal{A}_0(\Gamma(s)) \cap N_i \neq \emptyset$ and $\mathcal{A}_0(\Gamma(s)) \cap N_j \neq \emptyset$, $\mathcal{A}_0(\Gamma(s')) \subset (N_1 \cup N_2)$ for each $|s' - s| < \delta_s$,*

ii, *$\tilde{\pi}\mathcal{N}_0(\Gamma(s), \tilde{M}) \setminus (N_1 \cup N_2)$ is non-empty and totally disconnected;*

(H2), *For each $s' \in (s - \delta_s, s + \delta_s)$, $\Gamma(s')$ is equivalent to $\Gamma(s)$, namely, there exists a neighborhood of $\mathcal{N}_0(\Gamma(s))$, denoted by U , such that $\Gamma(s') - \Gamma(s) \in \ker i_U^*$.*

7.2. Variational construction. Given $x \in M$ and $c \in H^1(M, \mathbb{R})$, there exists at least a forward (backward) c -semi static curve $\gamma_c^+ : [0, \infty) \rightarrow M$ ($\gamma_c^- : (-\infty, 0] \rightarrow M$) such that $\gamma_c^\pm(0) = x$. It determines certain velocity $v_{x,c}^\pm = \dot{\gamma}_c^\pm(0)$.

The task in this subsection is to prove the following by variational method:

Theorem 7.1. *If c is connected to c' by a generalized transition chain, then*

1, *there exists an orbit of the Lagrange flow ϕ_L^t , $d\gamma : \mathbb{R} \rightarrow TM$ which connects the Aubry set $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$, namely, $\alpha(d\gamma) \subseteq \tilde{\mathcal{A}}(c)$ and $\omega(d\gamma) \subseteq \tilde{\mathcal{A}}(c')$;*

2, *given $x, x' \in M$ and arbitrarily small $\delta > 0$, there exists an orbit $(\gamma, \dot{\gamma})$ of ϕ_L^t passing through δ -neighborhood of the points $(x, v_{x,c}^+)$ and $(x', v_{x,c'}^-)$ successively, namely, $t < t'$ such that $(\gamma(t), \dot{\gamma}(t)) \in B_\delta(x, v_{x,c}^+)$ and $(\gamma(t'), \dot{\gamma}(t')) \in B_\delta(x', v_{x,c'}^-)$.*

Proof. Before starting the proof, we establish the following proposition:

Proposition 7.1. *Given an Aubry set, the Aubry distance from any Aubry class \mathcal{A}_i to all other Aubry classes is assumed have positive lower bound, namely, $d_c(\mathcal{A}_i, \mathcal{A}_j) \geq d > 0$ for all $j \neq i$. Let*

$$N_i = \{m \in M : d_c(m, x) \leq d/6, \forall x \in \mathcal{A}_i\},$$

then for all $m_0, m_1 \in N_i$ and for any $x \in \mathcal{A}_i$ one has

$$(7.1) \quad h^\infty(m_0, x) + h^\infty(x, m_1) = h^\infty(m_0, m_1);$$

for any $m_0, m_1 \in N_i$ and any $x \in \mathcal{A} \setminus \mathcal{A}_i$ one has

$$(7.2) \quad h^\infty(m_0, x) + h^\infty(x, m_1) \geq h^\infty(m_0, m_1) + \frac{d}{2}.$$

Proof. : For each pair of points $(m_0, m_1) \in M \times M$, we claim that there exists some Aubry class \mathcal{A}_j such that

$$h^\infty(m_0, m_1) = h^\infty(m_0, \xi) + h^\infty(\xi, m_1)$$

holds for each $\xi \in \mathcal{A}_j$. Indeed, let $k_i \rightarrow \infty$ be a subsequence of integers such that

$$\lim_{i \rightarrow \infty} h^{k_i}(m_0, m_1) = h^\infty(m_0, m_1)$$

let $\gamma^{k_i} : [-k_i, k_i] \rightarrow M$ be the minimizer for $h^{k_i}(m_0, m_1)$. There exists at least one point $\gamma^{k_i}(t_i)$ such that $d(\gamma^{k_i}(t_i), \mathcal{A}) \rightarrow 0$ as $k_i \rightarrow \infty$. Otherwise, the quantity $h^{k_i}(m_0, m_1) \rightarrow \infty$ as $k_i \rightarrow \infty$.

Given $m \in N_i$, we claim that (7.1) and (7.2) hold if $m_0 = m_1 = m$. Let $k_\ell \rightarrow \infty$ be a sequence such that $\lim_{k_\ell \rightarrow \infty} h^{k_\ell}(m, m) = h^\infty(m, m)$ and let $\gamma_m^{k_\ell}(t) : [-k_\ell, k_\ell] \rightarrow M$ be the minimizer of $h^{k_\ell}(m, m)$. There is a positive number $d' > 0$ such that the ordinary distance $d(\gamma_m^{k_\ell}(t), \mathcal{A}_j) \geq d' > 0$ for any $t \in [-k_\ell, k_\ell]$ and $j \neq i$. Otherwise along the curve $\gamma_m^{k_\ell}(t)$ there exists a point getting closer and closer to a point $x_j \in \mathcal{A}_j$, consequently, one would obtain from the property that $d_c(\mathcal{A}_i, \mathcal{A}_j) \geq d > 0$ for each $j \neq i$ that

$$\begin{aligned} h^\infty(m, m) &= h^\infty(m, x_j) + h^\infty(x_j, m) \\ &\geq h^\infty(x_i, x_j) - h^\infty(x_i, m) + h^\infty(x_j, x_i) - h^\infty(m, x_i) \\ &\geq \frac{5}{6}d \end{aligned}$$

where $x_i \in \mathcal{A}_i$. On the other hand, we have

$$h^\infty(m, m) \leq h^\infty(m, x_i) + h^\infty(x_i, m) \leq \frac{1}{6}d.$$

It is a contradiction. Therefore, some $x_i \in \mathcal{A}_i$ and $t_\ell \in [0, k_\ell]$ exist such that $t_\ell \rightarrow \infty$ as $k_\ell \rightarrow \infty$ and $\gamma_m^{k_\ell}(t_\ell) \rightarrow x_i$. This proves (7.1) in case $m_1 = m_2$.

Let us consider two different points $m_0, m_1 \in N_i$. For $x \in \mathcal{A}_j$ with $j \neq i$, we let $\zeta_u^k(t, m_0, x): [-k, k] \rightarrow M$ be the curve which minimizes the quantity $h^k(m_0, x)$, let k_j be the subsequence of k such that $\lim_{k_j \rightarrow \infty} h^{k_j}(m_0, x) = h^\infty(m_0, x)$. In autonomous case, it converges as $k \rightarrow \infty$. Similarly, we let $\zeta_s^k(t, x, m_1): [-k, k] \rightarrow M$ be the curve which minimizes the quantity $h^k(x, m_1)$, let k'_j be the sequence of k such that $\lim_{k'_j \rightarrow \infty} h^{k'_j}(x, m_1) = h^\infty(x, m_1)$. Let $\ell = 0, 1$, $\gamma_\ell^k: [-k, k] \rightarrow M$ be the minimizer of $h^k(m_\ell, m_\ell)$ and let k_ℓ be the subsequence of k such that $h^{k_\ell}(m_\ell, m_\ell) \rightarrow h^\infty(m_\ell, m_\ell)$. By the proof we just finished, there exists $x_\ell \in \mathcal{A}_i$ and integer $t_\ell^i \in [-k_\ell, k_\ell]$ such that $\gamma_\ell^{k_\ell}(t_\ell^i) \rightarrow x_\ell$ and $t_\ell^i \rightarrow \infty$ as $k_\ell \rightarrow \infty$. Let $\xi_{01}^k: [-k, k] \rightarrow M$ be the minimizer of $h^k(x_0, x_1)$, k_{01}^i be the subsequence of k such that $h^{k_{01}^i}(x_0, x_1) \rightarrow h^\infty(x_0, x_1)$, let $\xi_{10}^i: [0, k] \rightarrow M$ be the minimizer of $h^k(x_1, x_0)$, k_{10}^i be the subsequence of k such that $h^{k_{10}^i}(x_1, x_0) \rightarrow h^\infty(x_1, x_0)$. Given arbitrarily small $\delta > 0$, we have sufficiently large k_j , k'_j , k_0^i , k_1^i , k_{01}^i and k_{10}^i such that

$$\begin{aligned} |h^\infty(m_0, x) - h^{k_j}(m_0, x)| &< \delta, \\ |h^\infty(x, m_1) - h^{k'_j}(x, m_1)| &< \delta, \\ |h^\infty(m_\ell, m_\ell) - h^{k_\ell^i}(m_\ell, m_\ell)| &< \delta, \quad \ell = 0, 1 \\ |h^\infty(x_0, x_1) - h^{k_{01}^i}(x_0, x_1)| &< \delta, \\ |h^\infty(x_1, x_0) - h^{k_{10}^i}(x_1, x_0)| &< \delta. \end{aligned}$$

Since $x_0, x_1 \in \mathcal{A}_i$, we have $d_c(x_1, x_0) = 0$. Consequently,

$$\begin{aligned} (7.3) \quad & h^{t_0^i}(m_0, x_0) + h^{k_{01}^i}(x_0, x_1) + h^{k_{10}^i - t_1^i}(x_1, m_1) \\ & + h^{t_1^i}(m_1, x_1) + h^{k_{10}^i}(x_1, x_0) + h^{k_0^i - t_0^i}(x_0, m_0) \\ & \leq \frac{1}{3}d + 6\delta. \end{aligned}$$

Since x is in Aubry class \mathcal{A}_j , while $x_0, x_1 \in \mathcal{A}_i$, one has

$$\begin{aligned} (7.4) \quad & h^{k'_j}(x, m_1) + h^{t_1^i}(m_1, x_1) + h^{k_{10}^i}(x_1, x_0) \\ & + h^{k_0^i - t_0^i}(x_0, m_0) + h^{k_j}(m_0, x) \\ & \geq d - 5\delta. \end{aligned}$$

Because δ is arbitrarily small, by subtracting (7.3) from (7.4) we obtain

$$\begin{aligned} & h^\infty(m_0, x) + h^\infty(x, m_1) - \frac{2}{3}d \\ & \geq h^\infty(m_0, x_0) + h^\infty(x_0, x_1) + h^\infty(x_1, m_1) \\ & \geq h^\infty(m_0, m_1) \end{aligned}$$

it verifies (7.2). Since (7.2) holds for each $x \in \mathcal{A}_j$ with $j \neq i$ and for any $m_0, m_1 \in N_i$, (7.1) hold for each $x \in \mathcal{A}_i$ and for any $m_0, m_1 \in N_i$. This completes the proof of the proposition. \square

Let us begin the proof of Theorem 7.1. We only need to study the autonomous case. Time periodic case can be treated in the same way. Therefore, $\alpha(\Gamma(s)) \equiv E > \min \alpha$. By adding suitable constant on the Lagrangian, we assume $E = 0$ to simplify notation.

As the first step, let us choose a sequence of local connecting orbits $d\zeta_i$, which are successively connected in the sense that $\omega(d\zeta_i) \cap \alpha(d\zeta_{i+1}) \neq \emptyset$. The global connecting orbits are constructed shadowing such a sequence of orbits. According to the study in the previous sections, if s' is close to s , each Aubry set $\tilde{\mathcal{A}}(\Gamma(s))$ can be connected to some $\tilde{\mathcal{A}}(\Gamma(s'))$ by either type h , or type c locally minimal orbits. Thus, there is a sequence $0 = s_0 < s_1 < \cdots < s_k = 1$ such that for each $0 \leq j < k$, $\tilde{\mathcal{A}}(\Gamma(s_j))$ is connected to $\tilde{\mathcal{A}}(\Gamma(s_{j+1}))$ by some local minimal orbits.

Let $c_j = \Gamma(s_j)$, $j = 0, 1, \dots, k$. We divide the set $\{0, 1, \dots, k\}$ into m sets $\{0, 1, \dots, k\} = \{0, 1, \dots, i_1\} \cup \{i_1 + 1, \dots, i_2\} \cup \cdots \cup \{i_{m-1} + 1, \dots, i_m = k\}$. The rule to make such a partition is that for all $i = i_j, i_j + 1, \dots, i_{j+1} - 1$, $\tilde{\mathcal{A}}(c_i)$ is connected to $\tilde{\mathcal{A}}(c_{i+1})$ by a local minimal orbit of the same type. More precisely, let Λ_c and Λ_h be the subset of $\{i_1, i_2, \dots, i_m\}$, $\Lambda_c \cup \Lambda_h = \{i_1, i_2, \dots, i_m\}$, $\Lambda_c \cap \Lambda_h = \emptyset$. If $i_j \in \Lambda_i$, then for all $i = i_j, i_j + 1, \dots, i_{j+1} - 1$, $\tilde{\mathcal{A}}(c_i)$ is connected to $\tilde{\mathcal{A}}(c_{i+1})$ by a local minimal orbit of type i ($i = c$, or h).

Recall the definition of transition chain (Definition 7.1). Since the map $c \rightarrow \tilde{\mathcal{N}}(c, M)$ is upper semi-continuous, once the Mañé set $\tilde{\mathcal{N}}(\Gamma(s))$ is in the case (H1) (or H2), then for s' sufficiently close to s , the set $\tilde{\mathcal{N}}(\Gamma(s'))$ is also in the case (H1) (or H2). Thus, for each $i_j \in \Lambda_h$, by choosing c_{i_j-1} and c_{i_j+1} sufficiently close to c_{i_j} and c_{i_j+1-1} respectively, we can assume that both c_{i_j-1} and c_{i_j+1} satisfy the condition (H1) also.

With class c_i we associate an admissible coordinate system $x \rightarrow G_i^{-1}x$, write the inverse of G_i with $G_i^{-1} = [g_{i,1}^{-1}, g_{i,2}^{-1}, \dots, g_{i,n}^{-1}]^t$. Because we consider the problem on $H^{-1}(E)$ with $E > \min \alpha$, we can choose G_i for each $i \in \Lambda_c$ (see Proposition 6.3) such that, in new coordinates, the first component of $\omega(\mu_{c_i})$ is positive for each ergodic c_i -minimal measure. In virtue of the upper semi-continuity of Mañé set on cohomology class, one can assume that

$$\langle g_{j,1}^{-1}, \omega(\mu_{c_i}) \rangle > 0, \quad \forall j = i-1, i$$

holds for each ergodic component μ_{c_i} provided c_{i-1} is chosen suitably close to c_i . It means that the $\omega_1(\mu_{c_i}) > 0$ holds in the coordinates not only determined by G_i , but also determined by G_{i-1} as well as by G_{i+1} . Therefore, $\exists x_{i,1} > 0$ such that

$$(7.5) \quad \begin{aligned} \langle g_{i,1}^{-1}, \Delta \tilde{\gamma}_i \rangle &\geq 2\pi, \quad \text{whenever } \langle g_{i-1,1}^{-1}, \Delta \tilde{\gamma}_i \rangle \geq x_{i,1}, \\ \langle g_{i-1,1}^{-1}, \Delta \tilde{\gamma}_i \rangle &\geq 2\pi, \quad \text{whenever } \langle g_{i,1}^{-1}, \Delta \tilde{\gamma}_i \rangle \geq x_{i,1} \end{aligned}$$

holds for each c_i -semi-static curve γ , where $\tilde{\gamma}_i$ denotes its lift to universal covering space and $\Delta \tilde{\gamma}_i = \tilde{\gamma}_i(t') - \tilde{\gamma}_i(t)$ with $t' > t$.

As the second step, we describe the minimal properties of local connecting orbits of type- h as well as of type- c .

The case of type- h . For each integer $i \in \bigcup_{i_j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, the condition (H1) holds for generalized transition chain. Namely, in certain finite covering space $\tilde{\pi} : \tilde{M} \rightarrow M$, the Aubry set for i and $i + 1$ consists of more than one but

finitely many classes $\mathcal{A}(c_\ell, \check{M}) = \cup \mathcal{A}_{\ell,j}$ for $\ell = i, i+1$. By assumption of (H1), some open domains $N_i^-, N_{i+1}^+ \subset \check{M}$ exist such that $d(N_i^-, N_{i+1}^+) > 0$, $\mathcal{A}(c_i, \check{M}) \cap N_i^- \neq \emptyset$, $\mathcal{A}(c_{i+1}, \check{M}) \cap N_{i+1}^+ \neq \emptyset$. $\mathcal{A}(c_i, \check{M}) \subset N_i^- \cup N_{i+1}^+$ and $\mathcal{A}(c_{i+1}, \check{M}) \subset N_i^- \cup N_{i+1}^+$. $\mathcal{N}(c_i, \check{M}) \setminus N_i^- \cup N_{i+1}^+ \neq \emptyset$ is totally disconnected.

The new coordinate system $x \rightarrow G_i^{-1}x$ of \mathbb{T}^n is introduced such that (7.5) holds. If writing $\check{M} = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{T}\}$, we introduce the covering space $\bar{M} = \{(x_1, x_2, \dots, x_n) : x_1 \in \mathbb{R}, x_i \in \mathbb{T} \text{ for } i \geq 2\}$. We shall work with this covering space $\pi : \bar{M} \rightarrow \check{M}$. For closed 1-forms η_i and $\bar{\mu}_i$ on \bar{M} we use the same symbol to denote their natural lift to \bar{M} .

Recall the proof of Theorem 6.5. Some decomposition of \bar{M} exists such that $\bar{M} = \bar{M}_i^+ \cup \bar{M}_{i,0} \cup \bar{M}_i^-$ such that \bar{M}_i^+ is diffeomorphic to $[0, \infty) \times \mathbb{T}^{n-1}$, \bar{M}_i^- is diffeomorphic to $(-\infty, 0] \times \mathbb{T}^{n-1}$ and $\bar{M}_{i,0}$ is diffeomorphic to $(0, 1) \times \mathbb{T}^{n-1}$. Some open and connected disks $U_i^+, U_i^-, D_i, D'_i \subset \bar{M}_{i,0}$ and $\delta_i > 0$ exist such that $D_i + \delta_i \subset D'_i$, $(\pi D'_i + \delta_i) \cap N_i^- = \emptyset$ and $(\pi D'_i + \delta_i) \cap N_{i+1}^+ = \emptyset$, the intersection of any two of these sets is empty and the closure of $\bar{M}_i^+ \cup U_i^+ \cup D_i \cup U_i^- \cup \bar{M}_i^-$, denoted by \bar{M}_i^c , is connected,

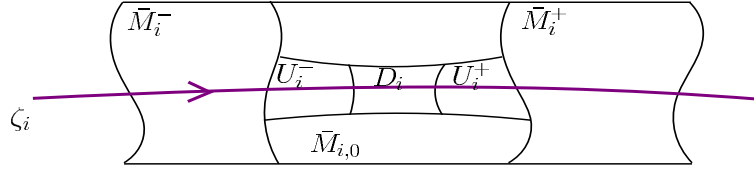


FIGURE 8.

As it was studied in the subsection of 6.3 (local connecting orbit of type- h), some function $w_i, \rho_i : \bar{M} \rightarrow [0, 1]$, some closed 1-form $\eta_i, \bar{\mu}_i, \varrho_i$ and some small constant $\delta_i > 0$ exist such that $[\eta_i] = c_i$, $[\bar{\mu}_i] = c_{i+1} - c_i$ and $\text{supp } \bar{\mu}_i \cap \bar{D}_i = \emptyset$, $\text{supp } w_i \subset D'_i$, $w_i|_{D_i} = \text{constant}$, $\rho_i(x) = 1$ if $x \in \bar{M}_i^+ \cup U_i^+$, $\rho_i(x) = 0$ if $x \in U_i^- \cup \bar{M}_i^-$.

Let $\mu_i = \rho_i \bar{\mu}_i$, $\psi_i = w_i \varrho_i$ we introduce a space-step Lagrangian

$$L_{\eta_i, \mu_i, \psi_i} = L - \eta_i - \mu_i - \psi_i.$$

In virtue of Theorem 6.5, some curve $\bar{\zeta}_i \in \mathcal{C}_{\eta_i, \mu_i, \psi_i}$ (pseudo connecting orbit set) such that $d\zeta_i = (\zeta_i, \dot{\zeta}_i)$ connects certain Aubry class $\tilde{\mathcal{A}}_{j_i}(c_i)$ to another Aubry class $\tilde{\mathcal{A}}_{j'_{i+1}}(c_{i+1})$, where $\zeta_i = \pi \bar{\zeta}_i$. Such curve stays entirely in the interior of \bar{M}_i^c . Therefore, along such curve both μ_i and ψ_i do not contribute to the Euler-Lagrange equation, consequently, $d\zeta_i$ is an orbit of ϕ_L^t .

As pointed out in Definition 6.4, such local connecting orbit of type- h possesses certain kind of local minimality. There exist two $(n-1)$ -dimensional disks V_i^- and V_{i+1}^+ with $\pi V_i^- \subset N_i^- \setminus \mathcal{A}_{j_i}(c_i)$, $\pi V_{i+1}^+ \subset N_{i+1}^+ \setminus \mathcal{A}_{j'_{i+1}}(c_{i+1})$, big numbers $T_i^+ > 0$, suitably small $d_i > 0$ and quite small $\epsilon_i^* > 0$ such that $\bar{\zeta}_i(-T_i^+) \in V_i^-$, $\bar{\zeta}_i(T_i^+) \in V_{i+1}^+$

and

$$\begin{aligned}
(7.6) \quad & \min \left\{ h_{c_i}^\infty(x^-, m_0) + h_{\eta_i, \mu_i, \psi_i}^T(\bar{m}_0, \bar{m}_1) + h_{c_{i+1}}^\infty(m_1, x^+) : \right. \\
& \quad \left. (\bar{m}_0, \bar{m}_1, T) \in \partial(V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i]) \right\} \\
& \geq \min \left\{ h_{c_i}^\infty(x^-, m_0) + h_{\eta_i, \mu_i, \psi_i}^T(\bar{m}_0, \bar{m}_1) + h_{c_{i+1}}^\infty(m_1, x^+) : \right. \\
& \quad \left. (\bar{m}_0, \bar{m}_1, T) \in V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i] \right\} + 5\epsilon_i^*,
\end{aligned}$$

where $x^- \in \alpha(d\zeta_i) \subseteq \mathcal{A}_{j_i}(c_i)$, $x^+ \in \omega(d\zeta_i) \subseteq \mathcal{A}_{j'_i}(c_{i+1})$. The disks V_i^- and V_{i+1}^+ are chosen so that $\bar{\zeta}_i$ intersects them transversally, for each $(\bar{m}_0, \bar{m}_1, T') \in V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i]$, the minimizer of $h_{\eta_i, \mu_i, \psi_i}^T(\bar{m}_0, \bar{m}_1)$, $\bar{\gamma}_i(t, \bar{m}_0, \bar{m}_1, T)$ has the property

$$(7.7) \quad \bar{\gamma}_i(t) \in \bar{M}_i^c \quad \forall t \in [-T, T].$$

Let ζ_{i-1} be the locally minimal curve generating an orbits $d\zeta_{i-1}$ connecting $\tilde{\mathcal{A}}(c_{i-1})$ to $\tilde{\mathcal{A}}(c_i)$. Denote by $\tilde{\mathcal{A}}_{j'_i}$ the Aubry class which contains the ω -limit set of $d\zeta_{i-1}$. It is possible that $\mathcal{A}_{j'_i}$ is different from \mathcal{A}_{j_i} which contains the α -limit set of $d\zeta_i$. Let us assume that the Aubry set consists of k_i classes. By the result in [CP], the subscript of Aubry classes can be rearranged such that some c_i -semi-static curve $\gamma_{i,j}$ exists such that $\alpha(d\gamma_{i,j}) \subset \tilde{\mathcal{A}}_j$ and $\omega(d\gamma_{i,j}) \subset \tilde{\mathcal{A}}_{j+1}$ for $j = 1, \dots, k_i \pmod{k_i}$. So some positive integer $k \leq k_i$ exists such that $j_i - j'_i = k \leq k_i \pmod{k_i}$. Let $d_i = \min d_c(\mathcal{A}_j(c_i), \mathcal{A}_{j'}(c_i))$.

We choose some $(n-1)$ -dimensional small disks $V_{i,j}^\pm$ with $j = j'_i, \dots, j_i$ such that $V_{i,j'_i}^+ = V_i^+$, $V_{i,j_i}^- = V_i^-$, $V_{i,j}^\pm$ is located within $N_{i,j}$, a small neighborhood of $\mathcal{A}_j(c_i)$ such that $d_c(m, x) \leq d_i/6$ holds for each $m \in N_{i,j}$ and each $x \in \mathcal{A}_j(c_i)$ (see Proposition 7.1 for the definition of $N_{i,j}$), $\gamma_{i,j}$ intersects $V_{i,j}^-$ as well as $V_{i,j+1}^+$ transversally. These curves also have locally minimal property similar to the form of (7.6):

$$\begin{aligned}
(7.8) \quad & \min \left\{ h_{c_i}^\infty(x^-, m_0) + h_{c_i}^T(m_0, m_1) + h_{c_i}^\infty(m_1, x^+) : \right. \\
& \quad \left. (\bar{m}_0, \bar{m}_1, T) \in \partial(V_{i,j}^- \times V_{i,j+1}^+ \times [T_{i,j}^+ - \tau_i, T_{i,j}^+ + \tau_i]) \right\} \\
& \geq \min \left\{ h_{c_i}^\infty(x^-, m_0) + h_{c_i}^T(m_0, m_1) + h_{c_i}^\infty(m_1, x^+) : \right. \\
& \quad \left. (\bar{m}_0, \bar{m}_1, T) \in V_{i,j}^- \times V_{i,j+1}^+ \times [T_{i,j}^+ - \tau_i, T_{i,j}^+ + \tau_i] \right\} + 5\epsilon_i^*,
\end{aligned}$$

where $x^- \in \mathcal{A}_j(c_i)$, $x^+ \in \mathcal{A}_{j+1}(c_{i+1})$, $T_{i,j}^+$ is the time such that $\gamma_{i,j}(2T_{i,j}^+) \in V_{i,j+1}^+$ if $\gamma_{i,j}(0) \in V_{i,j}^-$. As semi-static curves are totally disconnected, by assumption, $V_{i,j}^-$ and $V_{i,j+1}^+$ can be chosen so that any curve in $\mathcal{N}_{j,j+1}(c_i)$ does not touch the boundary of $V_{i,j}^-$ and of $V_{i,j+1}^+$.

Note that $h_c^\infty = \lim_{T \rightarrow \infty} h_c^T$ in the autonomous case [Fa1], we find from Proposition 7.1 that, for any $\epsilon_i^* > 0$, there exists $T_{i,j}^- = T_{i,j}^-(\epsilon_i^*) > 0$ such that

$$|h_{c_i}^T(m, m') - h_{c_i}^\infty(m, x) - h_{c_i}^\infty(x, m')| \leq \epsilon_i^*$$

holds for each $T \geq T_{i,j}^-$, each $m, m' \in N_{i,j}$ and $x \in \mathcal{A}_j(c_i)$.

Let $t_i^+ = t_{i,j'_i}^+ < t_{i,j'_i}^- < t_{i,j'_i+1}^+ \cdots < t_{i,j_i}^+ < t_{i,j_i}^- = t_i^-$, $2\Delta t_{i,j}^+ = t_{i,j+1}^+ - t_{i,j}^-$, $2\Delta t_{i,j}^- = t_{i,j}^- - t_{i,j}^+$. A curve $\gamma: [t_i^+, t_i^-] \rightarrow \check{M}$ is called *admissible* for $V_{i,j}^\pm$ if

$$(7.9) \quad \gamma(t_{i,j}^\pm) = x_{i,j}^\pm \in V_{i,j}^\pm, \quad \forall j'_i \leq j \leq j_i,$$

where $\Delta t_{i,j}^+$ and $\Delta t_{i,j}^-$ are chosen to satisfy the condition

$$(7.10) \quad T_{i,j}^+ - \tau_i \leq \Delta t_{i,j}^+ \leq T_{i,j}^+ + \tau_i$$

which is for the inequality (7.8), the condition

$$(7.11) \quad T_{i,j}^- + T_{i,j}^+ + \tau_i \leq \Delta t_{i,j}^- + \Delta t_{i,j}^+ \leq T_{i,j}^- + T_{i,j}^+ + 2\tau_i$$

which is set so that the minimal curve does not touch the boundary of $V_{i,j}^-$ provided it passes through $V_{i,j}^+$ at $t = t_{i,j}^+$ and through $V_{i,j+1}^+$ at $t = t_{i,j+1}^+$ and the condition

$$(7.12) \quad T_{i,j}^- + T_{i,j-1}^+ + \tau_{i-1} \leq \Delta t_{i,j-1}^- + \Delta t_{i,j-1}^+ \leq T_{i,j}^- + T_{i,j-1}^+ + 2\tau_{i-1}.$$

which is set so that the minimal curve does not touch the boundary of $V_{i,j}^+$ provided it passes through $V_{i,j-1}^-$ at $t = t_{i,j-1}^-$ and through $V_{i,j}^-$ at $t = t_{i,j}^-$. These conditions define non-empty set for $(\Delta t_{i,j}^+, \Delta t_{i,j}^-)$ if sufficiently large $T_{i,j}^\pm$ is well chosen.

We consider the minimum of the following action among all admissible curves:

$$h_{c_i}^{t_i^+, t_i^-}(x_i^+, x_i^-) = \inf_{\substack{\gamma(t_i^+) = x_i^+ \in V_{i,j'_i}^+ \\ \gamma(t_i^-) = x_i^- \in V_{i,j_i}^-}} \int_{t_i^+}^{t_i^-} (L - \eta_i)(d\gamma)dt.$$

Let $\gamma(t, t_i^\pm, x_i^\pm): [t_i^+, t_i^-] \rightarrow \check{M}$ be the minimizer of the action. If $t_{i,j}^- - t_{i,j}^+$ is sufficiently large, the minimizer is smooth at each $t_{i,j'_i}^- < t_{i,j'_i+1}^+ < \cdots < t_{i,j_i}^+$. Indeed, if $(x_{i,j}^-, x_{i,j+1}^+, \Delta t_{i,j}^+) \in \partial(V_{i,j}^- \times V_{i,j+1}^+ \times [T_{i,j}^+ - \tau_i, T_{i,j}^+ + \tau_i])$ holds for some $j'_i \leq j < j_i$, one obtains from (7.8) that

$$\begin{aligned} & h_{c_i}^{\Delta t_i^-}(x_{i,j}^+, x_{i,j}^-) + h_{c_i}^{\Delta t_i^+}(x_{i,j}^-, x_{i,j+1}^+) + h_{c_i}^{\Delta t_{i+1}^-}(x_{i,j+1}^+, x_{i,j+1}^-) \\ & \geq h_{c_i}^\infty(\xi, x_{i,j}^-) + h_{c_i}^{\Delta t_i^+}(x_{i,j}^-, x_{i,j+1}^+) + h_{c_i}^\infty(x_{i,j+1}^+, \zeta) \\ & \quad + h_{c_i}^\infty(x_{i,j}^+, \xi) + h_{c_{i+1}}^\infty(\zeta, x_{i,j+1}^-) - 2\epsilon_i^* \\ & \geq h_{c_i}^\infty(\xi, \hat{x}_{i,j}^-) + h_{c_i}^{\Delta t_i^+}(\hat{x}_{i,j}^-, \hat{x}_{i,j+1}^+) + h_{c_i}^\infty(\hat{x}_{i,j+1}^+, \zeta) \\ & \quad + h_{c_i}^\infty(x_{i,j}^+, \xi) + h_{c_{i+1}}^\infty(\zeta, x_{i,j+1}^-) + 3\epsilon_i^* \\ & \geq h_{c_i}^{\Delta t_i^-}(x_{i,j}^+, \hat{x}_{i,j}^-) + h_{c_i}^{\Delta t_i^+}(\hat{x}_{i,j}^-, \hat{x}_{i,j+1}^+) + h_{c_i}^{\Delta t_{i+1}^-}(\hat{x}_{i,j+1}^+, x_{i,j+1}^-) + \epsilon_i^* \end{aligned}$$

where $\xi \in \mathcal{A}_j(c_i)$, $\zeta \in \mathcal{A}_{j+1}(c_i)$, \bar{x}_i^- as well as \bar{x}_{i+1}^+ is the intersection point of a curve in $\mathcal{N}_{j,j+1}(c_i)$ with $V_{i,j}^-$ and with $V_{i,j+1}^+$ respectively. This contradicts the minimality of γ . If the minimizer $\gamma(t, t_i^\pm, x_i^\pm)$ is not smooth at $x_{i,j}^-$, we join the points $\gamma(t_{i,j}^- - \delta, t_i^\pm, x_i^\pm)$ and $\gamma(t_{i,j}^- + \delta, t_i^\pm, x_i^\pm)$ by the minimizer of

$$h_{c_i}^\delta(\gamma(t_{i,j}^- - \delta, t_i^\pm, x_i^\pm), \gamma(t_{i,j}^- + \delta, t_i^\pm, x_i^\pm)).$$

By the argument below, one can see that this minimizer pass through $V_{i,j}^-$. Thus, one obtains a curve γ' by replacing the segment of $\gamma(t, t_i^\pm, x_i^\pm)|_{t_{i,j}^- - \delta, t_{i,j}^- + \delta}$ by this

minimizer. Let t' be the time of this curve passing through $V_{i,j}^-$, clearly, $t_{i,j}^+ - t' \in (T_{i,j}^+ - \tau_i, T_{i,j}^+ + \tau_i)$, $\gamma'(t')$ does not touch the boundary of $V_{i,j}^-$. But this is absurd.

The case of type- c . For each integer $i \in \bigcup_{i_j \in \Lambda_c} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, there exist an admissible section Σ_{c_i} , a neighborhood U_i of $\mathcal{N}(c_i) \cap \Sigma_{c_i}$, two closed 1-forms η_i and $\bar{\mu}_i$ on M with $[\eta_i] = c_i$, $[\bar{\mu}_i] = c_{i+1} - c_i$ and $\text{supp } \bar{\mu}_i \cap U_i = \emptyset$. Correspondingly, an admissible coordinate system $x \rightarrow G_i^{-1}x$ on M is chosen such that in the new coordinates, one has a covering space $\pi_i: \bar{M}_i = \mathbb{R} \times \mathbb{T}^{n-1} \rightarrow M$, the set $\pi_i^{-1}\Sigma_{c_i}$ consists of infinitely many compact components $\Sigma_{c_i}^j = \Sigma_{c_i}^0 + j e_1 = \{x = (x_1 + j, x_2, \dots, x_n) : x \in \Sigma_{c_i}^0\}$ ($j \in \mathbb{Z}$). \bar{M} is separated by $\Sigma_{c_i}^0$ into upper part \bar{M}_i^+ and lower part \bar{M}_i^- . A smooth function $\rho_i: \bar{M} \rightarrow [0, 1]$ is constructed such that $\rho_i = 0$ for $x \in \bar{M}_i^- \setminus (\Sigma_{c_i}^0 + \delta_i)$ and $\rho_i = 1$ for $x \in \bar{M}_i^+ \setminus (\Sigma_{c_i}^0 + \delta_i)$. The number $\delta_i > 0$ is chosen so small such that $(\Sigma_{c_i}^0 + \delta_i) \cap (\mathcal{N}([\eta_i + \mu_i]) + \delta_i) \subset U_i$, (cf. formula (6.3)). Let $\mu_i = \rho_i \bar{\mu}_i$.

To make notation simpler, for each integer $i \in \bigcup_{i_j \in \Lambda_c} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, we let $\psi_i = 0$, $V_i^- = \{q_1 = -K_i\}$ and $V_{i+1}^+ = \{q_1 = K_i\}$ in the coordinate system $q = G_i^{-1}x$ (see the corollary 6.1 for the definition of K_i). Again, let

$$L_{\eta_i, \mu_i, \psi_i} = L - \eta_i - \mu_i - \psi_i.$$

Since the class c_i is equivalent to the class c_{i+1} and they are close to each other, one sees from Theorem 6.4 that each curve $\bar{\gamma} \in \mathcal{C}_{\eta_i, \mu_i, \psi_i}$ determines a locally minimal orbit type- c $d\gamma$ which is an orbit of ϕ_L^t and connects $\tilde{\mathcal{A}}(c_i)$ to $\tilde{\mathcal{A}}(c_{i+1})$.

Let $\bar{m} \in V_i^-$, $\bar{m}' \in V_{i+1}^+$ and let $\bar{\gamma}_i(t, \bar{m}, \bar{m}') : [-T, T] \rightarrow \bar{M}$ be the minimizer of

$$h_{\eta_i, \mu_i, \psi_i}^T(\bar{m}, \bar{m}') = \inf_{T' > 0} h_{\eta_i, \mu_i, \psi_i}^{T'}(\bar{m}, \bar{m}').$$

According to Lemma 6.2 and Corollary 6.1, $\exists K_i > 0$, $T_i^+ = T_i^+(K_i) > 0$, there exists $T < T_i^+$ such that $h_{\eta_i, \mu_i, \psi_i}^T(\bar{m}, \bar{m}') = \inf_{T' > 0} h_{\eta_i, \mu_i, \psi_i}^{T'}(\bar{m}, \bar{m}')$ provided the first coordinate of \bar{m} as well as of \bar{m}' satisfies the condition that $\bar{m}_1 \leq -K_i$ and $\bar{m}'_1 \geq K_i$. The minimizer $\bar{\gamma}_i(t, \bar{m}, \bar{m}') : [-T, T] \rightarrow \bar{M}$ satisfies

$$(7.13) \quad \bar{\gamma}_i(t, \bar{m}, \bar{m}') \in U_i, \quad \text{whenever } \bar{\gamma}_i(t, \bar{m}, \bar{m}') \in \Sigma_{c_i}^0 + \delta_i.$$

As the codimension one disks V_i^+ and V_i^- are codimension one tori in different coordinate systems, their relative position is fixed in the universal covering space. For this purpose, we define the following covering spaces:

$$\mathbb{R}^n \xrightarrow{\bar{\pi}_i} \bar{M}_i \xrightarrow{\pi_i} \check{M}_i,$$

where $\check{M}_i = \{(q_1, \dots, q_n) : q_i \bmod 2i_j\pi\}$ and $\bar{M}_i = \mathbb{R} \times \{(q_2, \dots, q_n) : x_i \bmod 2i_j\pi\}$ in the coordinate system $q = G_i^{-1}x$. For simplicity, we use the same notation for a fundamental domain of V_i^\pm in \mathbb{R}^n , namely, restricted on V_i^\pm the projection is a homeomorphism and $\bar{\pi}_i V_i^\pm = \check{V}_i^\pm$. In the coordinate system $q = G_i^{-1}x$, both V_i^- and V_{i+1}^+ are some translation of unit $(n-1)$ -cube $\{q_1 = 0, q_i \in [0, 1) \forall i = 2, \dots, n\}$. Clearly, there exist $(n-1)$ irreducible integer vectors $(v_2^i, v_3^i, \dots, v_n^i)$ such that

$$\bar{\pi}_i^{-1} V_i^- = \bigcup_{k_\ell \in \mathbb{Z}, \ell=2, \dots, n} V_i^- + k_\ell v_\ell^i, \quad \bar{\pi}_i^{-1} V_{i+1}^+ = \bigcup_{k_\ell \in \mathbb{Z}, \ell=2, \dots, n} V_{i+1}^+ + k_\ell v_\ell^i.$$

Given a fundamental domain V_i^+ and a c_i -semi static curve γ_i , one can find a lift of this curve to the universal covering space, denoted by $\tilde{\gamma}_i$, which intersects $V_i^+ \subset \mathbb{R}^n$. Clearly, it uniquely determines a fundamental domain V_i^- (up to a translation) by requiring the conditions that the curve $\tilde{\gamma}_i$ intersect it also, $\bar{\pi}_{i-1}V_i^-$ is above the $V_i^+ \subset \bar{M}_{i-1}$ and $\min\{q_1^- - q_1^+ : q^- \in \bar{\pi}_{i-1}V_i^-, q^+ \in V_i^+\} = K'_i$. We say that the two fundamental domains V_i^+ and V_i^- are (c_i, K'_i) -related if they satisfy these conditions.

Let V_i^+ and V_i^- be (c_i, K'_i) -related fundamental domains. Given positive integers k_i^+, k_i^- , we define

$$\mathbf{k}_i^\pm V_i^\pm = \bigcup_{|k_\ell| \leq k_i^\pm, \ell=2, \dots, n} V_i^\pm + k_\ell v_\ell^i.$$

Obviously, $\bar{\pi}_{i-1}\mathbf{k}_i^+ V_i^+ = V_i^+ \subset \bar{M}_{i-1}$. Let $\tilde{z}_i^+ \in \mathbf{k}_i^+ V_i^+$, $\tilde{z}_i^- \in \mathbf{k}_i^- V_i^-$ one defines the minimal action of L_{c_i} connecting these two points

$$h_{c_i}(\tilde{z}_i^+, \tilde{z}_i^-) = \inf_{T>0} \inf_{\substack{\tilde{\zeta}(-T)=\tilde{z}_i^+ \\ \tilde{\zeta}(T)=\tilde{z}_i^-}} \int_{-T}^T L_{c_i}(d\tilde{\zeta}(s))ds.$$

Let

$$h_{c_i}(\mathbf{k}_i^+ V_i^+, \mathbf{k}_i^- V_i^-) = \min_{\substack{\tilde{z}_i^+ \in \mathbf{k}_i^+ V_i^+ \\ \tilde{z}_i^- \in \mathbf{k}_i^- V_i^-}} h_{c_i}(\tilde{z}_i^+, \tilde{z}_i^-).$$

Clearly, for fixed k_i^+ , some positive number $\epsilon_i > 0$ and suitably large integer k_i^- exist such that $\mathbf{k}_i^+ V_i^+$ does not touch $\mathbf{k}_i^- V_i^-$,

$$(7.14) \quad h_{c_i}(\tilde{z}_i^+, \tilde{z}_i^-) > h_{c_i}(\mathbf{k}_i^+ V_i^+, \mathbf{k}_i^- V_i^-) + \epsilon'_i, \quad \text{if } d(\tilde{z}_i^-, \partial \mathbf{k}_i^- V_i^-) \leq 1.$$

Clearly, $h(\tilde{z}_i^+, \tilde{z}_i^-) \rightarrow \infty$ if $k_i^- \rightarrow \infty$ and $\tilde{z}_i^- \in \partial \mathbf{k}_i^- V_i^-$.

One can also define related fundamental domains V_i^- and V_{i+1}^+ similarly. Given a fundamental domain V_i^- and a curve $\gamma \in \mathcal{C}_{\eta_i, \mu_i}$, we pick up a lift of this curve to the universal covering space, denoted by $\tilde{\gamma}$, which intersects the section V_i^- . A unique fundamental domain V_{i+1}^+ exists where this curve intersects. Recall the projection of these two fundamental domains takes the for $V_{i+1}^+ = \{q_1 = K_i\}$ and $V_i^- = \{q_1 = -K_i\}$ in the configuration space \bar{M}_i . We say that the two fundamental domains V_i^- and V_{i+1}^+ are (η_i, μ_i, K_i) -related.

As the Lagrangian L_{η_i, μ_i} is clearly well-defined in the universal covering space, let us consider its action in the universal covering space:

$$h_{\eta_i, \mu_i}(\tilde{z}_i^-, k^* \tilde{z}_{i+1}^+) = \inf_{T>0} \inf_{\substack{\tilde{\zeta}(-T)=\tilde{z}_i^- \\ \tilde{\zeta}(T)=k^* \tilde{z}_{i+1}^+}} \int_{-T}^T L_{\eta_i, \mu_i}(d\tilde{\zeta}(s))ds,$$

where $k^* \tilde{z}_{i+1}^+ = \tilde{z}_{i+1}^+ + \sum_{\ell=2, \dots, n} k_\ell v_\ell^i$ stands for a translation and $k = (k_2, \dots, k_n)$. Obviously, one has

$$\inf_{k \in \mathbb{Z}^{n-1}} h_{\eta_i, \mu_i}(\tilde{z}_i^-, k^* \tilde{z}_{i+1}^+) = h_{\eta_i, \mu_i}(\tilde{z}_i^-, \tilde{z}_{i+1}^+) = \inf_{T>0} h_{\eta_i, \mu_i}^T(x_i^-, x_{i+1}^+)$$

where the term $\inf_{T>0} h_{\eta_i, \mu_i}^T(x_i^-, x_{i+1}^+)$ has been defined before by consider the action in the configuration space \bar{M}_i . As above, one defines

$$h_{\eta_i, \mu_i}(\mathbf{k}_i^- V_i^-, \mathbf{k}_{i+1}^+ V_{i+1}^+) = \min_{\substack{\tilde{z}_i^- \in \mathbf{k}_i^- V_i^- \\ \tilde{z}_{i+1}^+ \in \mathbf{k}_{i+1}^+ V_{i+1}^+}} h_{\eta_i, \mu_i}(\tilde{z}_i^-, \tilde{z}_{i+1}^+).$$

Again, for fixed k_i^- , some positive number $\epsilon_i > 0$ and suitably large k_{i+1}^+ exist such that $\mathbf{k}_i^- V_i^-$ does not touch $\mathbf{k}_{i+1}^+ V_{i+1}^+$ and

$$(7.15) \quad h_{\eta_i, \mu_i}(\tilde{z}_i^-, \tilde{z}_{i+1}^+) > h_{\eta_i, \mu_i}(\mathbf{k}_i^- V_i^-, \mathbf{k}_{i+1}^+ V_{i+1}^+) + \epsilon_i, \quad \text{if } d(\tilde{z}_{i+1}^+, \partial \mathbf{k}_i^+ V_{i+1}^+) \leq 1.$$

Clearly, $h_{\eta_i, \mu_i}(\tilde{z}_i^-, \tilde{z}_{i+1}^+) \rightarrow \infty$ if $k_{i+1}^+ \rightarrow \infty$ and $\tilde{z}_{i+1}^+ \in \partial \mathbf{k}_i^+ V_{i+1}^+$.

Let $\tilde{V}_i^\pm = \mathbf{k}_i^\pm V_i^\pm$. By induction, these sections \tilde{V}_i^\pm are well defined such that V_i^+ and V_i^- are (c_i, K'_i) -related, V_i^- and V_{i+1}^+ are (η_i, μ_i, K_i) -related, the formulae (7.14) and (7.15) are satisfied.

As the third step of the construction, let us clarify what conditions the candidates of minimal curve are required to satisfy.

Let $\gamma: [-K, K'] \rightarrow M$ be an absolutely continuous curve joining m to m' , i.e. $\gamma(-K) = m$ and $\gamma(K') = m'$. We divide the interval $[-K, K']$ into $2i_m + 1$ subintervals

$$[-K, K'] = [t_0^+, t_0^-] \cup [t_0^-, t_1^+] \cup \cdots \cup [t_{i_m}^+, t_{i_m}^-],$$

where $t_0^+ = -K$, $t_{i_m}^- = K'$. Correspondingly, we divide the curve into $2i_m + 1$ segments $\gamma_i^- = \gamma|_{[t_i^+, t_i^-]}$, $\gamma_i^+ = \gamma|_{[t_i^-, t_{i+1}^+]}$ for $i = 0, 1, 2, \dots, i_m - 1$, and $\gamma_{i_m}^- = \gamma|_{[t_{i_m}^+, t_{i_m}^-]}$.

We fix a lift $\tilde{\gamma}$ of γ to the universal covering space \mathbb{R}^n by choosing $\bar{\pi}G_0^{-1}\tilde{\gamma}(t_0^-) \in V_0^-$. Correspondingly, each γ_i^\pm has its lift $\tilde{\gamma}_i^\pm$ to \mathbb{R}^n .

The curve γ is required to satisfy the conditions:

1, for each $i = 0, 1, 2, \dots, i_m - 1$, there is some $k_i \in \mathbb{Z}$ such that

$$(7.16) \quad \begin{aligned} \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_i^-) - (2k_i \pi, 0, \dots, 0) &\in V_i^-, \\ \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_{i+1}^+) - (2k_i \pi, 0, \dots, 0) &\in V_{i+1}^+; \end{aligned}$$

2, for $i \in \bigcup_{j \in \Lambda_c} \{i_j + 1, \dots, i_{j+1} - 1\}$, $\tilde{\gamma}(t_i^\pm) \in \tilde{V}_i^\pm$. Let $\Delta t_i^+ = \frac{1}{2}(t_{i+1}^+ - t_i^-)$ and $\Delta t_i^- = \frac{1}{2}(t_i^- - t_{i-1}^+)$. To formulate the conditions for Δt_i^\pm , let us consider the quantity

$$h_{c_i}^{\Delta t}(\tilde{z}_i^+, \tilde{z}_i^-) = \inf_{\substack{\tilde{\xi}(-\Delta t) = \tilde{z}_i^+ \in \tilde{V}_i^+ \\ \tilde{\xi}(\Delta t) = \tilde{z}_i^- \in \tilde{V}_i^-}} \int_{-\Delta t}^{\Delta t} (L - \eta_i)(d\tilde{\xi}(t)) dt.$$

One obtains from the proof of Lemma 6.2 that $h_{c_i}^{\Delta t}(\tilde{z}_i^+, \tilde{z}_i^-) \rightarrow \infty$ as $\Delta t \rightarrow 0$ or $\rightarrow \infty$. Thus, if $T_i^- = T_i^-(\tilde{z}_i^+, \tilde{z}_i^-)$ is defined as the quantity such that $h_{c_i}^{T_i^-}(\tilde{z}_i^+, \tilde{z}_i^-) = \min_{\Delta t} h_{c_i}^{\Delta t}(\tilde{z}_i^+, \tilde{z}_i^-)$, then we find $0 < T_i^-(\tilde{z}_i^+, \tilde{z}_i^-) < \infty$. Since both $V_{i,-}^+$ and $V_{i,+}^-$ are compact, there exist $0 < \hat{T}_i^- < \check{T}_i^- < \infty$ such that $\hat{T}_i^- < T_i^-(\tilde{z}_i^+, \tilde{z}_i^-) < \check{T}_i^-$ holds for each $\tilde{z}_i^+ \in \tilde{V}_{i,-}^+$ and $\tilde{z}_i^- \in \tilde{V}_{i,+}^-$. Let

$$(7.17) \quad \Delta T_i^- = [\hat{T}_i^-, \check{T}_i^-].$$

The range of Δt_i^\pm is somehow implicitly defined. Let

$$(7.18) \quad \Delta T_i^+ = [T_i^+ - d_i, T_i^+ + d_i], \quad \forall i \in \bigcup_{i_j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$$

$$\Delta T_i^+ = (0, T_i^+], \quad \Delta T_i^- = [\hat{T}_i^-, \check{T}_i^-], \quad \text{for other } i \leq i_m.$$

See (7.6), (7.13) for the definition of T_i^+ and (7.17) for the definition of ΔT_i^- respectively.

The conditions for Δt_i^\pm are the following:

$$1, \Delta t_i^+ \in \Delta T_i^+ \text{ for all } 0 \leq i < i_m;$$

$$2, \Delta t_i^- \in \Delta T_i^- \text{ for } i \in \bigcup_{i_j \in \Lambda_c} \{i_j + 1, i_j + 2, \dots, i_{j+1} - 1\};$$

3, for $i \in \bigcup_{i_j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, as it has been assumed that the Aubry set $\mathcal{A}(c_i)$ contains finitely many classes, an orbit connects $\mathcal{A}(c_{i-1})$ to $\mathcal{A}(c_i)$ by approaching the Aubry class $\mathcal{A}_{j'_i}$ as $t \rightarrow \infty$, another orbit connects $\mathcal{A}(c_i)$ to $\mathcal{A}(c_{i+1})$ by approaching the Aubry class \mathcal{A}_{j_i} as the time retreat back to $-\infty$. For the time interval $[t_i^+, t_i^-]$, one has the partition

$$[t_i^+, t_i^-] = [t_i^+, t_{i,j'_i}^-] \cup [t_{i,j'_i}^-, t_{i,j'_i+1}^+] \cup \dots \cup [t_{i,j_i}^+, t_i^-],$$

and has restrictions for these quantities, formulae (7.10), (7.11), (7.12) and

$$(7.19) \quad T_{i,j_i}^- + T_i^+ + d_i \leq \Delta t_{i,j_i}^- + \Delta t_i^+ \leq T_{i,j_i}^- + T_i^+ + 3d_i,$$

$$(7.20) \quad T_{i+1,j'_{i+1}}^- + T_i^+ + d_i \leq \Delta t_{i+1,j'_{i+1}}^- + \Delta t_i^+ \leq T_{i+1,j'_{i+1}}^- + T_i^+ + 3d_i;$$

4, for $i = i_j$ with $i_j \in \Lambda_h$, by definition, c_i is equivalent to c_{i-1} , one has

$$(7.21) \quad \hat{T}_i^- + T_i^+ \leq \Delta t_i^+ + \Delta t_i^- \leq \check{T}_i^- + T_i^+ + 2d_i;$$

5, for $i = i_j$ with $i_j \in \Lambda_c$, by choosing c_{i-1} suitable close to c_i one can also assume that c_i is equivalent to c_{i-1} . Thus, one has

$$(7.22) \quad \hat{T}_i^- + T_{i-1}^+ \leq \Delta t_{i-1}^+ + \Delta t_i^- \leq \check{T}_i^- + T_{i-1}^+ + 2d_{i-1}.$$

As the system is autonomous, by choosing sufficiently large T_i^- , these conditions defines non-empty set for $(\Delta t_i^+, \Delta t_i^-)$.

Finally, let us introduce a modified Lagrangian and verify the smoothness of the minimizer of the action. Recall μ_i and ψ_i are defined on $\mathbb{R} \times \mathbb{T}^{n-1}$ in the coordinate system $q = G_i^{-1}x$, $G_i^*(\mu_i + \psi_i)(d\tilde{\gamma}) = (\mu_i + \psi_i)(\bar{\pi}G_i^{-1}d\tilde{\gamma})$ is well defined. We introduce a modified Lagrangian

$$L_{\eta_i, \mu_i, \psi_i} \rightarrow L - \eta_i - (k_i G_i)^*(\mu_i + \psi_i)$$

where k_i^* is a translation of q_1 : $(k_i)^*\phi(q, \dot{q}) = \phi(q_1 - 2\pi k_i, \hat{q}, \dot{q})$ on $T\bar{M}$ and the integer k_i is chosen so that (7.16) holds.

Let $\vec{t} = (t_0^-, t_1^\pm, \dots, t_{i_m-1}^\pm, t_{i_m}^+)$, $\vec{z} = (z_0^-, z_1^\pm, \dots, z_{i_m-1}^\pm, z_{i_m}^+)$. We consider the minimal action

$$(7.23) \quad \begin{aligned} h_L^{K,K'}(m, m', \vec{z}, \vec{t}) = & \inf \sum_{i=0}^{i_m} \int_{t_i^+}^{t_i^-} (L - \eta_i)(d\gamma_i^-(t)) dt \\ & + \sum_{i=0}^{i_m-1} \int_{t_i^-}^{t_{i+1}^+} (L - \eta_i - (k_i G_i)^*(\mu_i + \psi_i))(d\tilde{\gamma}_i^+(t)) dt \end{aligned}$$

where the infimum is taken over all C^1 -curves $\gamma: [-K, K'] \rightarrow M$ with fixed boundary conditions $\bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_i^-) - (2k_i \pi, 0, \dots, 0) = z_i^-$, $\bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_{i+1}^+) - (2k_i \pi, 0, \dots, 0) = z_{i+1}^+$ for $i = 0, 1, \dots, i_m - 1$, $\gamma(-K) = m$, $\gamma(K') = m'$ and satisfying the condition (7.16). Moreover, the restriction of γ on $[t_i^+, t_i^-]$ is admissible for the condition (7.9).

As the system is autonomous, the quantity $h_L^{K,K'}(m, m', \vec{z}, \vec{t})$ remains constant if (\vec{t}, K, K') is subject to a translation. Thus, it is a function of $K' - t_{i_m}^+$, $t_0^- + K$ and $\Delta \vec{t} = \{\Delta t_0^+, \Delta t_1^\pm, \dots, \Delta t_{i_m-1}^\pm\}$. Denote by $\Delta \vec{T}$ the domain where $\Delta \vec{t}$ takes its value. Let $\vec{V} = (V_0^-, V_1^\pm, \dots, V_{i_m-1}^\pm, V_{i_m}^+)$, where all entries have been well defined in the previous proof.

Denote by $\gamma(t; K, K', m, m', \vec{z}, \Delta \vec{t})$ the curve along which the quantity of (7.23) is realized, it obviously depends on the value $K, K', m, m', \vec{z}, \Delta \vec{t}$ and it may not be smooth at \vec{t} . Let $\Delta \vec{t}$ range over the set \vec{V} and $\Delta \vec{T}$, one obtains a minimizer. The purpose of the following steps is obviously to show that the minimizer is a solution of the Euler-Lagrange equation determined by L .

Let $h_L^{K,K'}(m, m')$ be the minimum of $h_L^{K,K'}(m, m', \vec{z}, \vec{t})$ over \vec{V} in \vec{z} and over $\Delta \vec{T}$ in $\Delta \vec{t}$ respectively:

$$h_L^{K,K'}(m, m') = \min_{\Delta \vec{t} \in \Delta \vec{T}, \vec{z} \in \vec{V}} h_L^{K,K'}(m, m', \vec{z}, \vec{t}),$$

denote the minimal curve by $\gamma(t; K, K', m, m')$, we claim that $d\gamma(t; K, K', m, m')$ is a solution of the Euler-Lagrange equation of L if K and K' are sufficiently large. To verify this claim, we need to show that

1, $d\gamma_i^+ = d\gamma|_{\Delta t_i^+}$ solves the Euler-Lagrange equation determined by L . Restricted on Δt_i^- , it obviously solves the Euler-Lagrange equation.

2, $\gamma(t; K, K', m, m')$ has no corner at $G_i z_i^-$ and $G_{i-1} z_i^+$ for each $i = 0, 1, \dots, i_m - 1$, i.e. it is smooth for the whole $t \in [-K, K']$. For each $i \in \Lambda_h$, as $\gamma_i^- = \gamma|_{\Delta t_i^-}$ is the minimizer for the curves admissible for the condition (7.9), it is smooth at each $t_{i,j'+1}^+ < \dots < t_{i,j_i}^+$.

Indeed, if $i \in \bigcup_{j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, we obtained from (7.7) that

$$\bar{\gamma}_i^+(t) \in U_i \quad \text{when} \quad \bar{\gamma}_{i,1}^+(t) - 2k_i \pi \in \Sigma_{c_i}^0 + \delta_i,$$

where $\bar{\gamma}_i^+ = \bar{\pi}_i G_i \gamma_i^+$. Since the support of $\bar{\mu}_i$ has no intersection with U_i and ψ_i is closed in U_i , while μ_i is closed and $\psi_i = 0$ in the region $\{\bar{\gamma}_{i,1}^+(t) - 2k_i \pi \notin [-\Delta_i, \Delta_i]\}$, the term μ_i and ψ_i have no contribution to the Euler-Lagrange equation along $\bar{\gamma}_i^+$.

For other i , the conclusion is obtained from (7.13) by similar argument. This proves the first one.

Recall the disks V_i^- and V_{i+1}^+ are defined in the covering space \bar{M}_i . We claim that γ does not touch the boundary of $G_i\pi_i V_i^- \times G_i\pi_i V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i]$ for $i \in \bigcup_{i_j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$. Let us assume the contrary, i.e. $(z_i^-, z_{i+1}^+, \Delta t_i^+) \in \partial(V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i])$ holds for some $i \in \bigcup_{i_j \in \Lambda_h} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, where $z_i^- = \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_i^-) - (2k_i\pi, 0, \dots, 0)$ and $z_{i+1}^+ = \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_i^+) - (2k_i\pi, 0, \dots, 0)$. Then, in $q = G_i^{-1}x$ -coordinates, we obtain from (7.6) and (7.18) that

$$\begin{aligned} & h_{c_i}^{\Delta t_i^-} (G_i^{-1} G_{i-1} \pi_i z_i^+, \pi_i z_i^-) + h_{\eta_i, \mu_i, \psi_i}^{\Delta t_i^+} (z_i^-, z_{i+1}^+) \\ & + h_{c_{i+1}}^{\Delta t_{i+1}^-} (\pi_i z_{i+1}^+, G_i^{-1} G_{i+1} \pi_i z_{i+1}^-) \\ & \geq h_{c_i}^\infty (\xi, \pi_i z_i^-) + h_{\eta_i, \mu_i, \psi_i}^{\Delta t_i^+} (z_i^-, z_{i+1}^+) + h_{c_{i+1}}^\infty (\pi_i z_{i+1}^+, \zeta) \\ & + h_{c_i}^\infty (G_i^{-1} G_{i-1} \pi_i z_i^+, \xi) + h_{c_{i+1}}^\infty (\zeta, G_i^{-1} G_{i+1} \pi_i z_{i+1}^-) - 2\epsilon_i^* \\ & \geq h_{c_i}^\infty (\xi, \pi_i \bar{x}_i^-) + h_{\eta_i, \mu_i, \psi_i}^{T_i^+} (\bar{x}_i^-, \bar{x}_{i+1}^+) + h_{c_{i+1}}^\infty (\pi_i \bar{x}_{i+1}^+, \zeta) \\ & + h_{c_i}^\infty (G_i^{-1} G_{i-1} \pi_i z_i^+, \xi) + h_{c_{i+1}}^\infty (\zeta, G_i^{-1} G_{i+1} \pi_i z_{i+1}^-) + 3\epsilon_i^* \\ & \geq h_{c_i}^{\Delta t_i^-} (G_i^{-1} G_{i-1} \pi_i z_i^+, \pi_i \bar{x}_i^-) + h_{\eta_i, \mu_i, \psi_i}^{T_i^+} (\bar{x}_i^-, \bar{x}_{i+1}^+) \\ & + h_{c_{i+1}}^{\Delta t_{i+1}^-} (\pi_i \bar{x}_{i+1}^+, G_i^{-1} G_{i+1} \pi_i z_{i+1}^-) + \epsilon_i^* \end{aligned}$$

where \bar{x}_i^- and \bar{x}_{i+1}^+ are the intersection points of a curve in $\mathcal{C}_{\eta_i, \mu_i, \psi_i}$ with V_i^- and with V_{i+1}^+ respectively, $\xi \in \mathcal{M}(c_{i-1})$ and $\zeta \in \mathcal{M}(c_i)$. This contradicts the minimality of γ , thus it verifies our claim.

To see the curve $\bar{\pi}_i G_i^{-1} \tilde{\gamma} - (2k_i\pi, 0, \dots, 0)$ is smooth at z_i^- , let us assume the contrary again. Let $z' = \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^-(t_i^- - \delta) - (2k_i\pi, 0, \dots, 0)$ and $z^* = \bar{\pi}_i G_i^{-1} \tilde{\gamma}_i^+(t_i^+ + \delta) - (2k_i\pi, 0, \dots, 0)$, here δ is chosen so small that $\Delta t_i^+ \pm \delta \in [T_i^+ - d_i, T_i^+ + d_i]$. This is possible since $(z_i^-, z_{i+1}^+, \Delta t_i^+) \notin \partial(V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i])$ implies that $T_i^+ - d_i < \Delta t_i^+ < T_i^+ + d_i$. We join these two points by a minimizer $\xi : [-\delta, \delta] \rightarrow M$ with $\xi(-\delta) = z'$ and $\xi(\delta) = z^*$

$$[A_{c_i}(\xi|_{[-\delta, \delta]})] = \inf_{\substack{\zeta(-\delta)=z' \\ \zeta(\delta)=z^*}} \int_{-\delta}^{\delta} (L - \eta_i)(d\xi(s)) ds.$$

If ξ passes through V_i^- , we obtain a curve γ' by replacing the segment of the minimizer $\gamma|_{[t_i^- - \delta, t_i^- + \delta]}$ with $G_i\xi : [-\delta, \delta] \rightarrow M$. Let $t_i'^-$ be the time for γ' passing through $G_i\pi V_i^-$, then $\frac{1}{2}(t_{i+1}^+ - t_i'^-) \in \Delta T_i^+$ and $\frac{1}{2}(t_i'^- - t_i^+) \in \Delta T_i^-$. Consequently, we would get an absolutely continuous curve which is admissible for each condition (see (7.18) and (7.20)). Along this curve we would get even smaller $h_L^{K, K'}(m, m')$, but this is absurd. So, we only need to show that ξ passes through $G_i V_i^-$. Indeed, as V_i^- is chosen small and transversal to the local connecting curve in $\mathcal{C}_{\eta_i, \mu_i, \psi_i}$, $\gamma_i^-(t)$ and $\gamma_i^+(t)$ approaches $G_i V_i^-$ from different sides as $t \searrow t_i^-$ and $t \nearrow t_i^-$ respectively. Otherwise, the minimality of γ would be violated. One refers to [BCV] for the details. The smoothness at $G_i z_{i+1}^+$ can be proved similarly.

The smoothness of γ at $t = t_i^\pm$ for $i \in \bigcup_{i_j \in \Lambda_c} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$ is obvious. Because of the formulae (7.14) and (7.15), $\tilde{\gamma}$ does not touch the boundary of \tilde{V}_i^\pm at the time of t_i^\pm respectively. Indeed, $\tilde{\gamma}$ approaches \tilde{V}_i^\pm from different sides as $t \searrow t_i^\pm$ and $t \nearrow t_i^\pm$ respectively. If $\tilde{\gamma}$ has a corner at $t = t_i^\pm$, let $\zeta: [t_i^\pm - \delta, t_i^\pm + \delta] \rightarrow \mathbb{R}^n$ be the minimizer of the action

$$A(\zeta|_{[t_i^\pm - \delta, t_i^\pm + \delta]}) = \inf_{\substack{\xi(t_i^\pm - \delta) = \tilde{\gamma}(t_i^\pm - \delta) \\ \xi(t_i^\pm + \delta) = \tilde{\gamma}(t_i^\pm + \delta)}} \int_{t_i^\pm - \delta}^{t_i^\pm + \delta} L_{c_i}(d\xi(s)) ds,$$

then the curve ζ penetrates through the disk \tilde{V}_i^\pm . Replacing $\tilde{\gamma}|_{[t_i^\pm - \delta, t_i^\pm + \delta]}$, a segment of $\tilde{\gamma}$ by this minimizer ζ one obtains a curve with smaller action. The contradiction verifies the smoothness.

If c_{i-1} is connected to c_i by a type- h orbit and $i \in \bigcup_{i_j \in \Lambda_c} \{i_j, i_j + 1, \dots, i_{j+1} - 1\}$, then V_i^+ is a small disk. By the same argument as above, one obtains the smoothness of $\tilde{\gamma}$ at $t = t_i^-$ and the smoothness at $t = t_i^+$ from the arguments for type- h .

As the system is autonomous, the following limit exists

$$h_L^\infty(m, m') = \lim_{K, K' \rightarrow \infty} h_L^{K, K'}(m, m').$$

We pick out a sequence of $\gamma(t; K, K', m, m')$ for large K and K' . Obviously, the set $\{\gamma(t; K, K', m, m')\}$ has at least one accumulation point $\gamma_\infty: \mathbb{R} \rightarrow M$ with the property $\alpha(d\gamma_\infty) \subseteq \tilde{\mathcal{A}}(c)$ and $\omega(d\gamma_\infty) \subseteq \tilde{\mathcal{A}}(c')$. As we have shown, it is an orbit of ϕ_L^t . This proves the first conclusion of the theorem.

For any two points $x, x' \in M$, we let $x = m$ and $x' = m'$. Clearly, the sequence $\{\gamma(t; K, K', x, x')|_{[0, K]}\}$ approaches to a forward c -semi static curve as $K \rightarrow \infty$, which starts from the point x , and $\{\gamma(t; K, K', x, x')|_{[t_{im}^-, K']}\}$ approaches to a backward c' -semi static curve as $K' \rightarrow \infty$, which approach to the point x' . Therefore, for sufficiently large K, K' , the initial value $(\gamma, \dot{\gamma})|_{t=0}$ falls into any prescribed δ -neighborhood of the points $(x, v_{x,c}^+)$ and the orbit arrives at the δ -neighborhood of $(x', v_{x,c'}^-)$ at the time $t = K'$. This completes the proof. \square

The proof for time-periodic system is similar, and a bit easier from technical point of view, since one can treat the time variable t as the first angle variable and take $\{t = 0\}$ the section for all classes. One does not need to introduce various coordinate systems $\{G_i^{-1}\}$ for different cohomology class. We omit the details here.

8. PROOF OF THE MAIN THEOREM

Once one obtains the existence of a generalized transition chain in the system (1.1), Theorem 1.1 is proved by applying Theorem 7.1. Therefore, the main purpose of this section is to show the genericity of such transition chains.

8.1. Candidate of transition chain. Let us consider the Hamiltonian (1.1). For any point $y_i \in h^{-1}(E)$ ($i = 0, 1, \dots, k$), there is a point $y'_i \in h^{-1}(E)$ such that

$|y'_i - y_i| < \delta$, the frequency $\omega'_i = \nabla h(y'_i)$ satisfying certain resonant condition, i.e. $\exists k_i \in \mathbb{Z}^3$ such that $\langle \omega'_i, k_i \rangle = 0$. Let

$$\Sigma_k = \{\omega \in \mathbb{R}^3 : \langle \omega, k \rangle = 0\}$$

denote the resonant plane determined by k , it passes through the origin. Note

$$\Omega_E = \{\omega = \nabla h(y) : y \in h^{-1}(E)\} \subset \mathbb{R}^3$$

is a surface homeomorphic to 2-sphere with the origin inside, $\Sigma_{k_i} \cap \Omega_E$ is a closed curve on that surface. In general, k_i is not co-linear with k_{i+1} . Thus, we obtain a resonant path connecting ω'_0 to ω'_k : moving from ω'_0 along the curve $\Sigma_{k_0} \cap \Omega_E$ to the intersection points with the curve $\Sigma_{k_1} \cap \Omega_E$, (the segment is denoted by Γ_{ω, k_0}), then moving along $\Sigma_{k_1} \cap \Omega_E$, passing through $\{\nabla h(y) : y \in B_\delta(y_1) \cap h^{-1}(E)\}$ and reaching the intersection point with $\Sigma_{k_2} \cap \Omega_E$ (the segment is denoted by Γ_{ω, k_1}) and so on, one obtains a resonant path

$$\Gamma_\omega = \Gamma_{\omega, 0} * \Gamma_{\omega, 1} * \cdots * \Gamma_{\omega, k-1}.$$

By construction, one sees that Γ_ω passes through each $\{\nabla h(y) : y \in B_\delta(y_i) \cap h^{-1}(E)\}$ with $i = 0, 1, \dots, k$. There are many resonant paths that pass through the neighborhood of each $\nabla h(y_i)$.

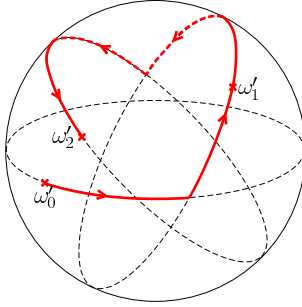


FIGURE 9. The resonant path in the surface of $h^{-1}(E)$.

Under the inverse of the frequency map $\omega \rightarrow y = (\nabla h)^{-1}(\omega)$ we obtain a path $\Gamma = \Gamma_0 * \Gamma_1 * \cdots * \Gamma_{k-1}$ in action-space, where $\Gamma_i = (\nabla h)^{-1}\Gamma_{\omega, i}$.

Let $\ell(\dot{x}) = \max_y (\langle \dot{x}, y \rangle - h(y))$ be the Lagrangian determined by the Hamiltonian h , ϕ_ℓ^t be the Lagrange flow. As the system is integrable, the action variable y keeps constant along each orbit of ϕ_ℓ^t which obviously lies in the support of certain c -minimal measure with $c = y$. In this sense, one obtains a path $\Gamma_c \subset H^1(\mathbb{T}^3, \mathbb{R})$ and $\Gamma_c = \Gamma$ if we identify $H^1(\mathbb{T}^3, \mathbb{R}) = \mathbb{R}^3$.

By the study of normal form, finitely many points $y_0 = y^0, y_1, \dots, y_N = y^1 \in \Gamma$ exist such that each $\omega_i = \nabla h(y_i)$ is rational frequency vector with period $T_i \leq K_0 \epsilon^{-\varrho}$ and

$$\bigcup_{0 \leq i \leq N} \{y : \|y - y_i\| \leq \mu T_i^{-1} \epsilon^\sigma\} \supset \Gamma + \frac{\mu}{2} \epsilon^{\sigma+\varrho},$$

where $\sigma < 1/6$, $\varrho = (1 - 3\sigma)/6$, see (A.9). Obviously, N depends on ϵ , the size of perturbation. Under $r - 2$ steps of KAM iteration, we obtain the normal form

$$H_i(x, y) = h(y) + \epsilon Z_{\epsilon, i}(x, y) + \epsilon R_{\epsilon, i}(x, y),$$

which is valid in the domain $\{y : \|y - y_i\| \leq \mu T_i^{-1} \epsilon^\sigma\} \times \mathbb{T}^3$. In which $Z_{\epsilon,i}$ is resonant with respect to ω_i , $\|R_{\epsilon,i}\|_{C^2} = O(\epsilon^{(k-2)\sigma})$.

As the term $Z_{\epsilon,i}$ is resonant with respect to ω_i , it can be written in the form

$$Z_{\epsilon,i}(x, y) = \sum_{\langle k, \omega_i \rangle = 0} Z_{\epsilon,i,k}(y) e^{i\langle k, x \rangle}.$$

Clearly, some indivisible $k_i \in \mathbb{Z}^3$ with $|k_i| \leq T_i$ exists such that $\langle k_i, \omega_i \rangle = 0$. Moreover, $\langle k', \omega_i \rangle = 0$ holds if $y_i \in \Gamma'$ or $\langle k^*, \omega_i \rangle = 0$ holds if $y_i \in \Gamma^*$. Clearly, an integer vector $k'_i \in \mathbb{Z}^3$ exists such that the matrix $I_i = (k', k_i, k'_i)$ or $I_i = (k^*, k_i, k'_i)$ is uni-module. Each $k \in \mathbb{Z}^n$ with $\langle k, \omega_i \rangle = 0$ determines uniquely an integer vector $(j_1, j_2) \in \mathbb{Z}^2$ such that $k = j_1 k_i + j_2 k'$ or $k = j_1 k_i + j_2 k^*$.

We introduce a coordinate transformation: $(x, y) \rightarrow (p, q)$ such that

$$(8.1) \quad q = I_i^t x, \quad p = I_i^{-1} y.$$

This coordinate transformation is symplectic and, with respect to q , $\bar{H}_i(p, q) = H(I_i^{-t} q, I_i p)$ is also a function defined in \mathbb{T}^n . Let y be the point where $\nabla h(y) = \omega$, then the gradient of $\bar{h}(p) = h(I_i p)$ satisfies

$$\bar{\omega}_i = \nabla \bar{h}(p) = (0, 0, \omega_{i3}),$$

and $\bar{Z}_{\epsilon,i}(p, q) = Z_{\epsilon,i}(I_i p, I_i^{-t} q)$ is independent of q_1 , thus we can write $\bar{Z}_{\epsilon,i}(p, \hat{q})$ if we write $\hat{q} = (q_1, q_2)$.

We still use (x, y) to denote the new coordinate system. Around a strong resonance point ω_i and the normal form takes the form

$$(8.2) \quad \bar{H}_i(x, y) = h(y) + \bar{Z}_{\epsilon,i}(\hat{x}, y) + \bar{R}_{\epsilon,i}(x, y)$$

in the new coordinate system (8.1), where $x = (x_1, x_2, x_3) = (\hat{x}, x_3)$, $y = (y_1, y_2, y_3) = (\hat{y}, y_3)$. This form remains valid in

$$\mathbb{T}^3 \times \{\|I_i(y - y_i)\| < T_i^{-1} \epsilon^\sigma\}$$

and at $y = y_i$ one has $\partial_y h = (0, 0, \omega_3)$ with $\omega_3 \neq 0$.

To make use of the results obtained in Section 4 and 5, let us consider the truncated Hamiltonian

$$H_{i,T} = h(y) + \epsilon Z_{\epsilon,i}(\hat{x}, y),$$

and the homogenized Hamiltonian

$$\bar{H}_i = \frac{1}{2} \langle A \hat{y} - \hat{y}_i, \hat{y} - \hat{y}_i \rangle + \epsilon V_i(\hat{x}),$$

where $V_i(\hat{x}) = Z_{0,i}(\hat{x}, y_i)$, $A = \partial_{\hat{y}}^2 h(y_i)$.

In the new coordinate system, we use $\Gamma_{\omega,i}$ to denote the resonant path in a neighborhood of $\omega_i = (0, 0, \omega_{i,3})$: $\omega \in \Gamma_\omega \iff I_i \omega \in \Gamma_{\omega,i}$. Recall the Fenchel-Legendre transformation $\mathcal{L}_\beta: H_1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$, determined by the β -function. Let β_h , $\beta_{H_{i,T}}$ and β_{H_i} be the β -function determined by h , $H_{i,T}$ and H_i respectively. Obviously, $\mathcal{L}_{\beta_h}(\Gamma_{\omega,i})$ is still a curve. As was studied in Subsection 4.3, $\mathcal{L}_{\beta_{H_{i,T}}}(\Gamma_{\omega,i})$ is composed by a flat \mathbb{F}_0 joined with two channels. See Figure 10 below. These channels are joined to the flat either at a point or along a sub-flat. The former case was thought difficult to handle.

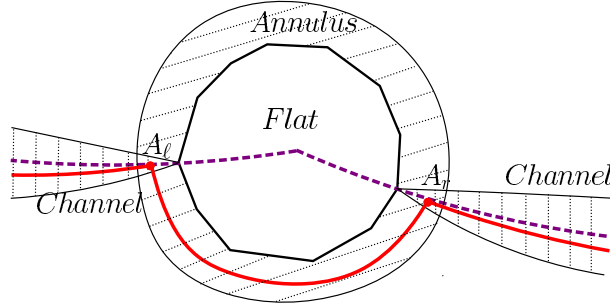


FIGURE 10. The transition chain under $\pi_3 : \alpha^{-1}(E) \rightarrow \mathbb{R}^2$, represented by the thick solid curve. Along the segment from A_ℓ to A_r , c_3 keeps constant. The dashed curve represents the curve \mathcal{L}_{β_h} .

8.2. Transition chain for incomplete intersection. Let \bar{L}_i be the Lagrangian determined by the truncated Hamiltonian \bar{H}_i , $\alpha_{\bar{L}_i}$ be the α -function. As it has been studied in the subsection 5.3, the double resonance corresponds to a flat $\mathbb{F}_0 \subset \alpha_{\bar{L}_i}^{-1}(E)$, around which there exists a annulus of incomplete intersection

$$\mathbb{A}' = \{(c_1, c_2, c_3) \in \alpha_{\bar{L}_i}^{-1}(E) : 0 < c_3 \leq \epsilon \Delta_0\}.$$

The following has been proved generic in Theorem 5.2. For each $c \in \mathbb{A}'$, the Mañé set does not cover the whole 3-torus. Let

$$N_{c,d_i} = \{x \in \mathbb{T}^3 : B_c(x) < d_i \epsilon\},$$

there exists some $d_i > 0$ such that N_{c,d_i} does not cover the whole 3-torus for each $c \in \mathbb{A}'$. Such results are obtained under the hypothesis **(H1~4)** proposed in the section 5.

As the truncated system is independent of x_3 ,

$$\pi_3 \mathcal{N}(c)|_\Sigma = \pi_3 \mathcal{N}(c)|_{\Sigma'}, \quad \pi_3 N_{c,d_i}|_\Sigma = \pi_3 N_{c,d_i}|_{\Sigma'}$$

where Σ as well as Σ' is a co-dimension one section on which x_3 keeps constant, $\pi_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the standard projection: $\pi_3(x_1, x_2, x_3) = (x_1, x_2)$. Indeed, let $G = G(\hat{x}, \hat{y})$ be the solution of $H(\hat{x}, \hat{y}, G) = \alpha(c)$, L_G be the Lagrangian determined by G , \mathcal{N}_{L_G} denote the Mañé set for L_G , one has $\pi_3 \mathcal{N}(c) = \mathcal{N}_{L_G}(\pi_3 c)$. If $\mathcal{N}(c)$ does not cover the 3-torus, $\pi_3 \mathcal{N}(c)$ does not cover the 2-torus. Because of $\alpha(c) > \min \alpha$, each c -minimal measure possesses non-zero rotation vector. Consequently, there exists some circle Σ_c^1 non-degenerately embedded into the 2-torus such that each c -minimal curve passes through Σ_c^1 transversally and the set $\pi_3 \mathcal{N}(c)|_{\Sigma_c^1}$ is topologically trivial, namely, some open intervals $I_j \subset \Sigma_c^1$ exist such that

$$\bigcup I_j \supset \pi_3 \mathcal{N}(c)|_{\Sigma_c^1}, \quad I_j \cap I_{j'} = \emptyset, \quad \forall j \neq j'.$$

One can choose suitable I_j so that

$$(8.3) \quad \bigcup I_j \times \{x_3 \in \mathbb{R} : \text{mod } 2\pi\} \supset N_{c,d_i}.$$

Let L_i be the Lagrangian determined by the Hamiltonian H_i . As $\|H_i - \bar{H}_i\|_{C^2} = O(\epsilon^{1+(r-2)\sigma})$, some $\epsilon_i > 0$ exists such that the Mañé set for L_i

$$(8.4) \quad \mathcal{N}(c) \subset N_{c,d_i}, \quad \forall \epsilon < \epsilon_i.$$

Let $\Gamma_{i,G_i} = \mathbb{A}_\epsilon \cap \{c_3 = G_i\}$ where

$$\mathbb{A}_\epsilon = \{(c_1, c_2, c_3) \in \alpha_{L_i}^{-1}(E) : 0 < c_3 \leq \epsilon \Delta_0\}.$$

It is a curve. With the preliminary works as above, some c -equivalence along this curve can be established. For each $c \in \Gamma_{i,G_i}$, let

$$\Sigma_c = \Sigma_c^1 \times \{x_3 \in \mathbb{R} \bmod\}.$$

By construction, each c -semi static curve passes through the section Σ_c transversally. Recall

$$V_c = \bigcap_U \{i_{U*} H_1(U, \mathbb{R}) : U \text{ is a neighborhood of } \mathcal{N}(c) \cap \Sigma_c\},$$

one sees that $c' - c \in V_c^\perp$ provided c' is close to c , $c'_3 = c_3$ and $\alpha(c') = \alpha(c)$, namely, $c' \in \Gamma_{i,G_i}$. In this case, some open set $U \supset \mathcal{N}(c) \cap \Sigma_c$ such that $V_c = i_{U*} H_1(U, \mathbb{R}) = \text{span}\{(0, 0, 1)\}$, from which one obtains that $V_c^\perp = \text{span}\{(1, 0, 0), (0, 1, 0)\}$. For each class $c' \in \Gamma$ very close to c , one has $c' - c = (\Delta c_1, \Delta c_2, 0) \in V_c^\perp$, thus, there exists a closed 1-form $\bar{\mu}$ such that $[\bar{\mu}] = c' - c$ and

$$\text{supp } \bar{\mu} \cap \mathcal{N}(c) \cap \Sigma_c = \emptyset.$$

Thus, any two classes along the curve Γ_{i,G_i} is equivalent, thus a transition chain for incomplete intersection is established, see Figure 10, the thick solid curve from the point A_ℓ to the point A_r .

8.3. Transition chain for complete intersection. By the study in the subsection 5.2 (see Theorem 4.2), there are two wedge-shaped channels \mathbb{W} and \mathbb{W}' which extend into the annulus \mathbb{A} , $\mathbb{W} = \cup_{\lambda \geq \lambda_0 > 0} \mathcal{L}_\beta(\lambda g)$ and $\mathbb{W}' = \cup_{\lambda \geq \lambda'_0 > 0} \mathcal{L}_\beta(\lambda g')$. Each of these channels corresponds to a normally hyperbolic cylinder $\bar{\Pi}_{\epsilon,g}$ invariant for the Lagrange flow: for each class c in the channel, $\tilde{\mathcal{N}}(c) \subset \bar{\Pi}_{\epsilon,g}$ if $c_3 \geq D\epsilon^{1+d'}$. The coordinate system is chosen so that $\dot{x}_3 > 0$ for each point in the domain we are interested in. The intersection of $\bar{\Pi}_{\epsilon,g}$ with $\{x_3 = \text{constant}\}$ is a 2-dimensional cylinder, denoted by $\Pi_{\epsilon,g}$ which is invariant for the return map. Indeed, to study the transition chain in this wedged-shaped channel, it is good enough to study the Lagrange flow determined by G , where $G(x, y, -\tau)$ solves the equation $H(x, y, \tau, G) = E_0$.

Each invariant circle in the cylinder is a Lipschitz curve. Indeed, in this case, the minimal measure is supported on a two-dimensional torus suspended by this invariant circle. Since the cylinder is normally hyperbolic, this torus has its local stable and unstable manifold. It implies that the forward (backward) weak KAM solution is differentiable when it is restricted in a neighborhood of this 2-torus. Note that weak KAM is a viscosity solution. It is well-known that a C^1 viscosity solution for Tonelli Hamiltonian must be $C^{1,1}$. Thus, one sees the Lipschitz property from the facts that stable (unstable) manifold is the graph of the differential of certain weak KAM, the cylinder is smooth and the invariant circle is the intersection of the cylinder with these stable (unstable) manifold.

A segment of the cylinder Π , bounded by two invariant circles, is invariant for the return map which preserves some “area” element. Let $\psi: \Pi_0 = [0, 1] \times \mathbb{T} \rightarrow \Pi$ be the map, it pulls back the standard closed 2-form $\omega = dx \wedge dy$ to a 2-form on Π . Since the second de Rham cohomology of a cylinder is trivial, by Moser’s theorem on the

isotopy of symplectic forms, there exists a diffeomorphism ψ_1 which transform this form to the standard 2-form, namely

$$(\psi \circ \psi_1)^* \omega = d\theta \wedge dI.$$

Since return map Φ_G preserves the form ω , one has

$$((\psi \circ \psi_1)^{-1} \circ \Phi_G \circ (\psi \circ \psi_1))^* d\theta \wedge dI = d\theta \wedge dI.$$

Some finite covering manifold $\tilde{\pi}: \check{M} \rightarrow \mathbb{T}^2$ is chosen so that the lift of the Aubry set on each section $\{x_3 = \text{constant}\}$ consists of two invariant circles, denoted by $\mathcal{A}_0(c, \check{M} \times \mathbb{T}) = \Upsilon_{1,c} \cup \Upsilon_{2,c}$. Let $N_{1,c}$ be a small neighborhood of Υ_1 and N_2 be a small neighborhood of Υ_2 such that N_1 is disjoint with N_2 . Obviously,

$$\mathcal{N}_0(c, \check{M}) \setminus (N_1 \cup N_2) \neq \emptyset.$$

To construct transition chain in this situation, one need to show it is totally disconnected set. The counterpart in the energy level of the autonomous system is totally disconnected orbits connecting two 3-dimensional cylinders.

One can find the argument for this issue in [CY1]. For convenience of reader, let us briefly describe here what is the idea of the proof. Let $u_{i,c}^\pm$ be the forward (backward) weak KAM solution determined by $\Upsilon_{i,c}$ with $(i = 1, 2)$, A point $x \in \mathcal{N}_0(c, \check{M})$ if and only if

$$x \in \arg \min(u_{1,c}^- - u_{2,c}^+) |_{\tau=0}, \quad \text{or} \quad x \in \arg \min(u_{2,c}^- - u_{1,c}^+) |_{\tau=0}.$$

Thus, the problem turns out to be another version: $\arg \min(u_{1,c}^\pm - u_{2,c}^\mp) |_{\tau=0}$ is totally disconnected when it is restricted in $\check{M} \setminus (N_1 \cup N_2)$. For this purpose, a regularity result in [CY1, CY2] is used:

Lemma 8.1. *For each of the weak KAM solutions under consideration, the Aubry set is assumed to be an invariant curve in a normally hyperbolic cylinder. Let \mathfrak{U}^\pm be the set of these weak KAM solutions in C^0 -function space, then this set has finite box dimensions*

$$D_B(\mathfrak{U}^\pm) \leq 3.$$

Proof. By Lemma 6.4 in [CY2], some “area” element $\sigma = d\theta \wedge dI$ is introduced so that each cohomology class c uniquely corresponds to an area σ and

$$\|u_{c,\sigma}^\pm - u_{c',\sigma'}^\pm\|_{C^0} \leq C(\sqrt{|\sigma - \sigma'|} + |c - c'|).$$

As c is restricted on a line, one immediately obtains the result from the definition of box dimension. \square

Let $S = I_1 \times I_2$ be a small rectangle such that $S \subset \check{M} \setminus (N_1 \cup N_2)$ and $\pi_{\dot{x}} \phi_G^k(x, \dot{x}) \notin S$ for each $0 \neq k \in \mathbb{Z}^-$ provided $x \in S$ and $\dot{x} = \partial_x u_\sigma^-(x)$, it makes sense as weak KAM is differentiable almost everywhere. Let $\pi_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the standard projection $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$.

Lemma 8.2. *It is an open-dense condition for G such that*

$$\pi_i \arg \min(u_{1,c}^- - u_{2,c}^+) |_{S \times \{\tau=0\}} \subsetneq I_i$$

holds for all c .

Proof. Since the system is positive definite in y , it has a generating function $W(x, x')$ such that

$$y' = \partial_{x'} W(x, x'), \quad y = -\partial_x W(x, x'),$$

if $(x', y') = \Phi_G(x, y)$. One has relation between the generating function and the Lagrangian. For \bar{x} and \bar{x}' in the universal covering space, we let

$$W_c(\bar{x}, \bar{x}') = \inf_t \inf_{\substack{\xi(0)=\bar{x} \\ \xi(t)=\bar{x}'}} \int_0^t G_c(d\xi(s)) ds + t\alpha(c),$$

and

$$W_c(x, x') = W(x, x') - \langle c, x' - x \rangle.$$

Let $x' = x'(x, y)$ be the function solving the second equation, one obtains somehow more explicit form of the map:

$$y' = \partial_{x'} W(x, x'(x, y)), \quad x' = x'(x, y).$$

If perturbing the generating function by $W(x, x') \rightarrow W(x, x') + W_1(x')$ where W_1 is a smooth function such that $\text{supp} W_1 \subset S$, one find the map will be

$$y' = \partial_{x'} W(x, x'(x, y)) + \partial_{x'} W_1(x'(x, y)), \quad x' = x'(x, y)$$

if $x' \in S$.

By the choice of S , $\gamma_{c,x}^\pm(\pm k) \notin S$ with $0 \neq k \in \mathbb{Z}^+$ if $\gamma_{c,x}^\pm$ is a forward (backward) c -semi-static curve such that $\gamma_{c,x}^\pm(0) = x$. Therefore, for the perturbed system the barrier function subjects a translation when it is restricted on S

$$B_c(x) \rightarrow B_c(x) + W_1(x), \quad \forall x \in S.$$

Given a function $B_c(x)$, we define $\pi_1^* B_c(x_1) = \min_{x_2} B_c(x_1, x_2)$. Because of Lemma 8.1, the box dimension of the set $\{\pi_1^* B_c\}$ is obviously not bigger than 3. Let \sim denote an equivalence in C^0 , $u \sim v$ if and only if $u - v = \text{constant}$. Let C^0 / \sim denote the quotient space inheriting the topology from C^0 . Therefore, in any 4-dimensional ball $\mathfrak{B}_{4,\delta} \subset C^0 / \sim$ there is an open-dense set $\mathfrak{P}_{4,\delta}$ such that

$$0 \notin \{B_c + W_1\}, \quad \forall W_1 \in \mathfrak{P}_{4,\delta}.$$

Such small perturbation of generating function can be realized by a small perturbation of the Hamiltonian.

Let Φ' be the map determined by the generating function $W + \kappa W_1$, where $\kappa = 0$ if $|x - x'|$ is sufficiently large and $\kappa = 1$ in the domain where we are interested in. The symplectic diffeomorphism $\Psi = \Phi' \circ \Phi^{-1}$ is close to identity. We choose a smooth function ρ with $\rho(0) = 0$ and $\rho(1) = 1$, let Φ'_s be the symplectic map determined by $W + \rho(s)\kappa W_1$ and let $\Psi_s = \Phi'_s \circ \Phi^{-1}$. Clearly, Ψ_s defines a symplectic isotopy between identity map and Ψ . Thus there is a unique family of symplectic vector field $X_s: T^*M \rightarrow TT^*M$ such that

$$\frac{d}{ds} \Psi_s = X_s \circ \Psi_s.$$

By the choice of perturbation, there is a simply connected and compact domain D_K such that $\Psi_s|_{T^*M \setminus D_K} = \text{id}$. It follows that there is a Hamiltonian $G_1(x, y, s)$ such that $dH_1(Y) = dx \wedge dy(X_s, Y)$ holds for any vector field Y . Re-parametrizing s by t we can make H_1 smoothly and periodically depend on t . To see that dG_1 is also small, let us make use of a theorem of Weinstein [W]. A neighborhood of the identity in the symplectic diffeomorphism group of a compact symplectic manifold \mathbf{M} can be identified with a neighborhood of the zero in the vector space of closed 1-forms on

M. Since Hamiltomorphism is a subgroup of symplectic diffeomorphism, there is a function G' , sufficiently close to G , such that $\Phi_{G_1} \circ \Phi_G = \Phi_{G'}^t|_{t=1}$. \square

Theorem 8.1. *It is a generic condition for G that for each $c \in \mathbb{W}$ with $\alpha_G \geq D\epsilon^{1+d'}$ the set*

$$\mathcal{N}_0(c, \check{M}) \setminus (N_1 \cup N_2) \neq \emptyset.$$

is totally disconnected.

Proof. Divide the region into countably many small rectangles and take the intersection of countable many open-dense set. \square

8.4. Criterion for strong and weak resonance. It is natural to ask, along the path Γ , how many many double resonant points need to be treated as strong resonance, for each given perturbation $\epsilon P(x, y)$. For each $\omega \in \Gamma$, the resonance condition

$$\langle k, \omega \rangle = 0$$

is always satisfied and at each double resonant point some other $k' \in \mathbb{Z}^3$ exists such that k' is linearly independent of k and

$$\langle k', \omega \rangle = 0.$$

Recall the process of KAM iteration, the main part of the resonant term is obtained by averaging the perturbation over a circle determined by these two resonant relations. It takes the form

$$Z = Z_k(\langle k, x \rangle, y) + Z_{k,k'}(\langle k, x \rangle, \langle k', x \rangle, y)$$

where

$$Z_k = \sum_{j \in \mathbb{Z} \setminus \{0\}} P_{jk}(y) e^{j \langle k, x \rangle i}, \quad Z_{k,k'} = \sum_{(j,l) \in \mathbb{Z}^2, l \neq 0} P_{jk+lk'}(y) e^{(j \langle k, x \rangle + l \langle k', x \rangle) i}.$$

Note P is C^r -function, there is estimate on the coefficient $P_{jk+lk'}$

$$|P_{jk+lk'}| \leq 8\pi^3 \|P\|_{C^r} \|jk + lk'\|^{-r},$$

which deduces the estimation

$$(8.5) \quad \|Z_{k,k'}\|_2 \leq d \|P\|_{C^r} \|k'\|^{-r+2}$$

where $d = d(k)$ depends on k . The function Z_k is periodic in $q = \langle k, x \rangle$, the following hypotheses is obviously open and dense:

(H1.1): For each $y \in \Gamma$, Z_k is non-degenerate at it maximal point, i.e. $\partial_{qq}^2 Z_k(q) > 0$ holds provided q is a maximal point.

Proof. Approximating $Z_k(y, q)$ by a function which is real analytical and periodic in q , one only needs to consider the case that $Z_k(y, q)$ is real analytical. In this case, the set

$$M_K = \bigcup_y \arg \max_q Z_K,$$

consists of finitely many segments of curve, joined together at finitely many points. Obviously, it is open dense that each of these curves is the graph of certain smooth function of q , not necessarily defined for whole $q \in \mathbb{T}$, M_K contains finitely many points, denoted by (q_i, y_i) , where the second derivative in q is equal to zero. Thus,

one can also assume that both q - and y -coordinates of these points are different. Writing the function in Taylor expansion

$$Z_K(q, y) - Z_K(q_i, y_i) = \sum_{i=1}^4 a_i(y)(q - q_i)^i + O((q - q_i)^5)$$

where $a_i(y_i) = 0$ for $i = 1, 2, 3$. It is open and dense that $a_4(y_i) < 0$, the derivative of a_i at $y = y_i$ does not equal to zero. Introducing a small perturbation $\Delta(q, y)$ such that $\Delta(q, y) = \delta(q - q_i)(y - y_i)$ when it is restricted in a small neighborhood of (q_i, y_i) . Given a suitably small neighborhood of (q_i, y_i) , by choosing sufficiently small $\delta \neq 0$ one has

$$\bigcup_y \arg \max_q (Z_K + \Delta) \cap \{(q, y) : \partial_{qq}(Z_K + \Delta) = 0\} = \emptyset$$

holds in the small neighborhood of (q_i, y_i) . \square

Given some Z_k satisfying the hypothesis **(H1.1)**, one has $\lambda > 0$ such that for each $y \in \Gamma$, $\partial_{qq}^2 Z_k \geq \lambda$ holds at the maximal point. Assume at $y' \in \Gamma$, the second resonant condition $\langle k', \omega(y') \rangle = 0$ is also satisfied. One thus obtains the normal form (8.2), by performing the coordinate transformation (8.1), obtains further the homogenized form of the truncated Hamiltonian

$$\bar{G} = \langle A\hat{y}, \hat{y} \rangle + V_k(x_2) + V_{k,k'}(\hat{x})$$

where \bar{G} solves the equation $\bar{H}(\hat{x}, \hat{y}, \bar{G}) = E > \min \alpha$. The Hamiltonian flow determined by $\langle A\hat{y}, \hat{y} \rangle + V_k(x_2)$ admits an invariant cylinder $\Pi_{k,k'}^0 = \{(\hat{y}, x_2) = \text{constant}\} \times \mathbb{T}$ which is normally hyperbolic. Applying the theorem of normally hyperbolic manifold, one find from the estimate (8.5) that some positive number $d_1 = d_1(\lambda) > 0$ exists such that $\Phi_{\bar{G}}^t$ also admits a normally hyperbolic and invariant cylinder $\Pi_{k,k'}$ close to $\Pi_{k,k'}^0$ provided

$$(8.6) \quad \|k'\|^{r-2} \geq \frac{d}{d_1} \|P\|_{C^r}.$$

It is a criterion to see whether the double resonance is thought as weak resonance and can be treated in the way for *a priori* unstable system. There are finitely many $k' \in \mathbb{Z}^3$ not satisfying this condition, thus are treated as strong double resonance.

Therefore, once a perturbation P is chosen such that **(H1.1)** is satisfied, there are finitely many double resonant frequencies are needed to treat as strong double resonance.

For strong double resonance, the smallest eigenvalue may be contributed by the term $Z_{k,k'}$. It is also open and dense condition that

(H1.2): at each strong double resonance point, the maximal point of $Z_k + Z_{k,k'}$ is non-degenerate, namely, the two eigenvalues of the Hessian matrix are positive $\lambda_{k,j} > 0$ for $j = 1, 2$ and not equal. Indeed, there exists $\nu > 0$ such that $\lambda_{k,2} \geq \nu \|k\|^{r-2}$ and $\lambda_{k,1} \geq \nu \|k\|^{r-2} \|k'\|^{r-2}$.

8.5. Proof of the main theorem. Given y_0, y_1, \dots, y_k we have chosen a resonant path Γ_ω so that $\mathcal{L}_{\beta_h}(\Gamma_\omega)$ passes through δ -neighborhood of each y_i . Let ϵP satisfy all hypothesis above, for convenience for reader, we formulate them here again:

(H1): for each strong double resonance, the potential V_i attains its maximum at one point only, the Hessian matrix of V_i at that point is negative definite. All eigenvalues are different: $-\lambda_2 < -\lambda_1 < 0 < \lambda_1 < \lambda_2$. (see **H1** in Subsection 5.1, **H1.1** and **H1.2** in Subsection 8.4);

(H2): for the Hamiltonian flow $\Phi_{\bar{G}_i}^t$, the stable manifold of the fixed point intersects the unstable manifold transversally along each minimal homoclinic orbit. Each minimal homoclinic orbit approaches to the fixed point along the direction Λ_1 : $\dot{\gamma}(t)/\|\dot{\gamma}(t)\| \rightarrow \Lambda_{x_1}$ as $t \rightarrow \pm\infty$. (see **H2** in Subsection 5.1. Here, \bar{G}_i solves the equation $H_i(x, y, \bar{G}_i) = E$, the transversality is in the sense that, at the intersection points, the tangent space of the stable and unstable manifold span the tangent space of the energy level.)

(H3): For each $c \in \partial^*\mathbb{F}_{0,i}$, the Aubry set does not contain minimal curve homoclinic to the origin (fixed point). (see **H2** in Subsection 5.1, each strong double resonance is related to a flat $\mathbb{F}_{0,i}$ corresponding to the Hamiltonian \bar{G}_i .)

(H4): For $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, there are finitely many $\theta_i \in \mathbb{R}$ such that, for each rotation vector $\theta_i g$, the Mather set consists of two periodic orbits, for other rotation vector θg , the Mather set consists of one periodic orbit only. All these periodic orbits are hyperbolic. (see **H4** in Subsection 5.2, also formulated for the Hamiltonian \bar{G}_i .)

(H5): For each $c \in \partial^*\mathbb{F}_{0,i}$ there exists a disk disjoint either with the support of μ_c or of μ , restricted on which, the set $\arg \min(U_c^- - U_c^+)$ is non-empty. (see **H5** in Subsection 5.3, also formulated for the Hamiltonian \bar{G}_i .)

Along the resonant path Γ_ω , the strong double resonance points are denoted by $\omega_0, \omega_1, \dots, \omega_m$, where the number m depends on P . Each flat $\mathcal{L}_{\beta_H}(\omega_i)$ is surrounded by an annulus $\mathbb{A}'_i \subset \alpha_H^{-1}(E)$. For each segment of Γ_ω connecting ω_i to ω_{i+1} , denoted by $\Gamma_{\omega,i}$, $\mathcal{L}_{\beta_H}(\Gamma_{\omega,i})$ constitutes a channel connecting \mathbb{A}'_i to \mathbb{A}'_{i+1} .

Divide $[0, 1]$ into $2m + 1$ intervals

$$[0, 1] = [0 = s_{0,i}, s_{0,c}] \cup [s_{0,c}, s_{1,i}] \cup \dots \cup [s_{m-1,i}, s_{m,c}] \cup [s_{m,c}, s_{m,i} = 1]$$

and let $\Gamma_{j,c}: [s_{j,i}, s_{j,c}] \rightarrow \alpha_H^{-1}(E)$, $\Gamma_{j,i}: [s_{j,c}, s_{j+1,i}] \rightarrow \alpha_H^{-1}(E)$ denote curves such that $\Gamma_{j,c}(s_{j,c}) = \Gamma_{j,i}(s_{j,c})$, $\Gamma_{j,i}$ falls into the annulus \mathbb{A}'_j along which c_3 keeps constant in the coordinate system used for the normal form and $\Gamma_{j,c}$ falls into the channel $\mathcal{L}_{\beta_H}(\Gamma_{\omega,j})$ connecting \mathbb{A}'_i to \mathbb{A}'_{i+1} . Let $\Gamma_{0,c}(0) \in \mathcal{L}_{\beta_H}(\omega_s)$ and $\Gamma_{m,c}(1) \in \mathcal{L}_{\beta_H}(\omega_e)$. The subscript “c” is used to indicate complete intersection and the subscript “i” denotes the incomplete intersection. We choose the chain as the conjunction of these curves

$$(8.7) \quad \Gamma = \Gamma_{0,c} * \Gamma_{0,i} * \dots * \Gamma_{m-1,i} * \Gamma_{m,c}.$$

Under the hypothesis (**H1~5**), the chain has been shown transitive, in the sense of Definition 7.1, when it is restricted on the segments $\Gamma_{j,i}$ ($j = 0, 1, \dots, m-1$). To guarantee it also transitive when it is restricted on each segment $\Gamma_{j,c}$ ($j = 0, 1, \dots, m$), one need to impose some condition which has been proved to be generic in Subsection 8.3, Theorem 8.1:

(H6): if the Aubry set covers certain 2-torus in \mathbb{T}^3 for $c \in \Gamma_{j,c}$, then certain finite covering manifold \tilde{M} and certain two-dimensional section Σ_c exist such that

$$\mathcal{N}(c, \tilde{M})|_{\Sigma_c} \setminus (\mathcal{A}(c, \tilde{M}) + \delta)|_{\Sigma_c} \neq \emptyset.$$

is totally disconnected.

Under these hypothesis, namely (H1~6), the path Γ defined in (8.7) is a transition chain. Choose sufficiently many $c_i \in \Gamma$ ($i = 0, 1, \dots, i_m$) such that

1, each $\tilde{\mathcal{A}}(c_i)$ is connected to $\tilde{\mathcal{A}}(c_{i+1})$ by local minimal orbit either of type- c or of type- h ;

2, among these classes, some classes c_{i_j} ($j = 0, 1, \dots, k$) exist such that c_{i_j} is very close to y_j (the prescribed action variables in Theorem 1.1) if one identifies both c_{i_j} and y_j as point in \mathbb{R}^3 .

Recall the proof of Theorem 7.1. Let $\gamma: [-K, K'] \rightarrow \mathbb{T}^3$ be the minimizer of the action (7.23) satisfying the boundary conditions $\gamma(-K) = x_0$ and $\gamma(K') = x_k$. Dividing $[-K, K']$ into $2i_m + 1$ intervals

$$[-K, K'] = [t_0^+, t_0^-] \cup [t_0^-, t_1^+] \cup \dots \cup [t_{i_m}^+, t_{i_m}^-],$$

imposing some constraints on γ at $t = t_i^\pm$ and conditions on sufficiently large $t_{i+1}^+ - t_i^-$ and $t_i^- - t_i^+$, one then proves that γ is a solution of the Lagrange equation determined by H . γ determines an orbit of the Hamiltonian flow Φ_H^t :

$$x(t) = \gamma(t), \quad y(t) = \frac{\partial L}{\partial \dot{x}}(\gamma(t), \dot{\gamma}(t)).$$

To see that this orbit visits the ball $B_\delta(x_0, y_0), B_\delta(x_k, y_k) \subset \mathbb{T}^3 \times \mathbb{R}^3$ and the balls $B_\delta(y_i) \subset \mathbb{R}^3$ ($i = 1, \dots, k-1$), we use the following properties:

Lemma 8.3. *Let $\gamma_{c,x,L}^-: (-\infty, 0] \rightarrow M$ be backward c -semi static curve for the Lagrangian L such that $\gamma_{c,x,L}^-(0) = x$. The set $V_{c,x,L}^- \subset T_x M$ is defined so that for each $v \in V_{c,x,L}^-$ there exists a backward c -semi static curve $\gamma_{c,x,L}^-$ such that $v = \dot{\gamma}_{c,x,L}^-(0)$. Then $L \rightarrow V_{c,x,L}^-$ is upper-semi continuous. The same is true for forward semi-static curves.*

As h is integrable, any backward (forward) c -semi static curve is c -static for all $t \in \mathbb{R}$. Along any c -minimal curve the action variable always take the value c . As the perturbation $h \rightarrow h + \epsilon P$ is very small, $\dot{\gamma}(-K)$ is very close to $V_{x,c_0,L}^+$ provided $t_0^- - t_0^+$ is sufficiently large. It follows that $\|y(-K) - y_0\|$ is very small provided ϵ is sufficiently small. In the same principle, one can see that $\|y(K') - y_k\|$ is also very small. Note the Aubry set is upper-semi continuous with respect to the Lagrangian. At $t_i = (t_i^- + t_i^+)/2$, $(\gamma(t_i), \dot{\gamma}(t_i))$ is very close to $\tilde{\mathcal{A}}(c_i)$, consequently, $\|y(t_i) - y_i\|$ is very small provided $|t_i^- - t_i^+|$ is sufficiently large. This proves that the Hamiltonian flow Φ_H^t admits an orbit that visits these balls in turn.

To complete the proof of Theorem 1.1, we only need to show the generic property. As the number of strong double resonant points is independent of the size of ϵ , some open-dense set $\partial\mathfrak{D} \subset \partial\mathfrak{B}_1$ exists, for each $P \in \partial\mathfrak{D}$ some $\epsilon_P > 0$ is associated such that the Hamiltonian flow Φ_H^t satisfies the conditions H1~5 provided $\epsilon \leq \epsilon_P$. From

the proof of Theorem 8.1 in the subsection 8.3, the condition **H6** is required for the intersection of countably many open-dense set contained in \mathfrak{B}_1 : $\cap_i \mathfrak{D}_i$.

Let $\mathfrak{D} = \{\lambda f : f \in \partial \mathfrak{D}, \lambda \in [0, 1]\}$. Clearly, each $\mathfrak{D} \cap \mathfrak{D}_i$ is open-dense \mathfrak{P}_i in the following sense: there exists an open-dense set $\partial \mathfrak{P}_i \subset \partial \mathfrak{B}_1$, for each $P \in \partial \mathfrak{P}_i$ some open dense set $I_P \subset [0, 1]$ exists such that $\lambda P \in \mathfrak{D} \cap \mathfrak{D}_i$ provided $\lambda \in I_P$. Take countably intersection of $\mathfrak{D} \cap \mathfrak{D}_i$ we obtain the generic condition stated in the main theorem (Theorem 1.1). This completes the proof of the main theorem.

APPENDIX A. NORMAL FORM

In this section, we study normal form of a nearly integrable Hamiltonian, from which one can get some geometric information about the related Mather sets as well as Mañé sets. In this section, the system is assumed to have arbitrary n -degrees of freedom

$$H(x, y, t) = h(y) + P_\epsilon(x, y, t), \quad (x, y, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}.$$

The perturbation can be autonomous as well as time-1-periodically non-autonomous. As t can be treated as the $(n+1)$ -th angle coordinate, we replace n by $n+1$ when we consider time-periodically dependent perturbation $P_\epsilon(x, y, t)$. Thus, we consider autonomous Hamiltonian only in this section.

A.1. KAM iteration at strong resonance. Let $\omega(y) = \nabla h(y)$ denote the frequency vector of the unperturbed system. For autonomous case, a frequency ω is called rational of (minimal) period T if $T\omega \in \mathbb{Z}^n$ and $t\omega \notin \mathbb{Z}^n$ for each $t \in (0, T)$.

If the frequency ω is rational of period T , with a function $g(x, y)$ on the torus one associates its time average $[g]$ along the orbits of the linear flow defined by ω : $x \rightarrow x + \omega t$

$$[g](x, y) = \frac{1}{T} \int_0^T g(x + \omega t, y) dt.$$

In terms of [Lo], we shall say that g is *resonant* (with respect to ω) if $g = [g]$, which implies that g is constant along the orbits of the linear flow $(x, y) \rightarrow (x + \omega t, y)$.

Let $B_R \subset \mathbb{R}^n$ be the ball of radius R around the origin, then there are positive numbers $M = M(R) \geq m = m(R) > 0$ such that

$$m\|v\|^2 \leq \langle \nabla^2 h(y)v, v \rangle \leq M\|v\|^2, \quad \forall y \in B_R, v \in \mathbb{R}^n.$$

Let σ and ϱ denote positive number such that

$$\sigma < \frac{1}{3}, \quad K = K(\epsilon) = K_0 \epsilon^{-\varrho}, \quad \varrho = \frac{1}{3}(1 - 3\sigma),$$

the value of σ will be specified later to satisfy certain covering property.

Denoted by $\{\omega_\lambda : \lambda \in \Lambda_{K,R}\} \subset B_{MR}$ the set of frequencies which are rational of period T with $T \leq K$. Clearly, Λ_K is a finite index set. Let $y_\lambda = \nabla^{-1}h(\omega_\lambda)$.

Let $i = (i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n$, i.e. i_j is non-negative $\forall j \in \{1, 2, \dots, n\}$. Let $|i| = \sum_{j=1}^n i_j$, $Y_i(y) = \prod_{j=1}^n y_j^{i_j}$.

Theorem A.1. *For a nearly integrable Hamiltonian $H(x, y, t) = h(y) + P_\epsilon(x, y, t)$ we assume that both h and P_ϵ are C^k -smooth with $k \geq 4$, and $\|P_\epsilon\|_{k, B_R \times \mathbb{T}^{n+1}} \leq \epsilon$. Some small $\epsilon_0 = \epsilon_0(M, m, n, k) > 0$ exists so that for each $\epsilon \leq \epsilon_0$ and each $y \in B_R$ there exist $y_\lambda \in B_R$ ($\lambda \in \Lambda_{K, R}$) such that $\|y - y_\lambda\| \leq \mu T^{-1} \epsilon^\sigma$ with $\omega_\lambda = \omega(y_\lambda)$ being rational of period T with $T \leq K$, and a canonical transformation \mathcal{F}_λ well defined on*

$$D_{y_\lambda, \epsilon} = \{(x, y) \in \mathbb{T}^n \times \mathbb{R}^n : \|y - y_\lambda\| \leq \mu T^{-1} \epsilon^\sigma\}$$

which reduce the Hamiltonian into the normal form

$$(A.1) \quad H \circ \mathcal{F}_\lambda(x, y, t) = h(y) + Z(x, y, t) + R(x, y, t)$$

where Z is resonant with respect to ω_λ and R is a higher order term with respect to Z in the following sense: we can write R in the form

$$R = \sum_{|i|=0}^{k-2} Y_i(y - y_\lambda) R_i(x, y),$$

and the following estimates hold:

$$\|Z\|_2 \leq 2\epsilon, \quad \|R_i\|_2 \leq \epsilon^{1+(k-2-i)\sigma} \quad \text{in } D_{y_\lambda, \epsilon} \times \mathbb{T}.$$

Proof. Such a canonical transformation is obtained by the composition of finite steps of the following KAM iteration.

Lemma A.1. *Let $H(x, y)$ be a Hamiltonian on $\mathbb{T}^n \times B_{\delta_r}$ with the form*

$$(A.2) \quad H(x, y, t) = h(y) + Z_r(x, y) + R_{r,0}(x, y) + R_{r,1}(x, y),$$

where Z_r is resonant with respect to $\omega = \nabla h(0)$, a rational frequency with period T , $Z_r, R_{r,0} \in C^{k-r}$, $R_{r,1} \in C^{k-r-1}$ ($k \geq r+2$) with

$$Z_r = \sum_{|i|=0}^{r-1} Y_i(y) Z_r^i(x, y),$$

$$R_{r,j} = \sum_{|i|=0}^{r+j} Y_i(y) R_{r,j}^i(x, y), \quad (j = 0, 1)$$

Let ϵ_r , η_{r+j} and δ_{r+j} be small positive numbers, such that

$$(A.3) \quad \|Z_r^i\|_{k-r} \leq \epsilon_r \delta_r^{-|i|}, \quad \|R_{r,j}^i\|_{k-r-j} \leq \frac{1}{1+j} \eta_{r+j} \delta_{r+j}^{-|i|},$$

for $j = 0, 1$, where the norm is defined on $\mathbb{T}^n \times B_{\delta_r}$. We assume that δ_{r+1} satisfies the condition

$$(A.4) \quad E_r T \delta_r^{-1} \eta_r < \delta_r - \delta_{r+1},$$

where $E_r = \frac{1}{2}(1 + 2r + \sum_{j=1}^r j(j+1))$, then there exists a canonical transformation \mathcal{F}_r which maps $\mathbb{T}^n \times B_{\delta_{r+1}}$ into $\mathbb{T}^n \times B_{\delta_r}$ and

$$\|\mathcal{F}\|_r \leq ET \sum_{|i|=0}^r \delta_r^{|i|} \eta_{r,|i|,0} \quad \text{in } \mathbb{T}^n \times B_{\delta_{r+1}}$$

where $E = E(n, r)$ depends on n and r . If we write $H' = H \circ \mathcal{F}_r$ in the form of (A.2) with r being replaced by $r+1$

$$H' = H \circ \mathcal{F}_r = h + Z_{r+1} + R_{r+1,0} + R_{r+1,1},$$

let Z_{r+1}^i , $R_{r+1,0}^i$ and $R_{r+1,1}^i$ play the same role of Z_r^i , $R_{r,0}^i$ and $R_{r,1}^i$, the following holds in $\mathbb{T}^n \times B_{\delta_{r+1}}$ for certain positive number $D_r = D_r(n, r, h)$

$$(A.5) \quad \begin{aligned} \|Z_{r+1}^i\|_{k-r-1} &\leq (\epsilon_r + \eta_r) \delta_{r+1}^{-|i|}, \\ \|R_{r+1,0}^i\|_{k-r-1} &\leq D_r (T \eta_r \delta_r + (\epsilon_r + \eta_r) \eta_r \delta_r^{-1}) \delta_{r+1}^{-|i|}, \\ \|R_{r+1,1}^i\|_{k-r-2} &\leq 2D_r T^2 \eta_r^2 \delta_r^{-1-|i|}. \end{aligned}$$

Proof. Such a coordinate transformation \mathcal{F}_r is defined as the time-1-map of $\phi_{W_r}^t$, the Hamiltonian flow determined by the generating function W_r

$$W_r(x, y) = \sum_{|i|=0}^r Y_i(y) W_r^i(x, y)$$

where

$$W_r^i(x, y) = -\frac{1}{T} \int_0^T R_{r,0}^i(x + \omega s, y) s ds,$$

which is the solution of the following equation

$$\left\langle \omega, \frac{\partial W}{\partial x} \right\rangle = -R_{r,0}^i + [R_{r,0}^i].$$

Clearly, it satisfies

$$\|W_r^i\| \leq \frac{T}{2} \|R_{r,0}^i\|$$

for any translation-invariant norm $\|\cdot\|$ defined on the space of measurable functions. Obviously, \mathcal{F}_r is a map close to identity. If we write it in the form

$$\mathcal{F}_r : (x, y) \Rightarrow (x + U_r(x, y), y + V_r(x, y))$$

then

$$(U_r, V_r) = \int_0^1 \left(\frac{\partial W_r}{\partial y}, -\frac{\partial W_r}{\partial x} \right) \circ \phi_{W_r}^t dt,$$

from which and (A.3) we easily find that \mathcal{F}_r which maps $\mathbb{T}^n \times B_{\delta_{r+1}}$ into $\mathbb{T}^n \times B_{\delta_r}$.

By the standard Taylor formula, we have

$$H \circ \mathcal{F} = H + \{H, W\} + \int_0^1 (1-t) \{\{H, W\}, W\} \circ \phi_W^t dt.$$

Thus, we have

$$\begin{aligned} Z_{r+1} &= Z_r + [R_{r,0}], \\ R_{r+1,0} &= \left\langle \nabla h(p) - \omega, \frac{\partial W_r}{\partial x} \right\rangle + \{Z_r + R_{r,0}, W_r\} + R_{r,1} \circ \mathcal{F}_r, \\ R_{r+1,1} &= \int_0^1 (1-t) \{\{h + Z_r + R_{r,0}, W_r\}, W_r\} \circ \phi_{W_r}^t dt. \end{aligned}$$

To write Z_{r+1} , $R_{r+1,j}$ in the following form

$$\begin{aligned} Z_{r+1} &= \sum_{|i|=0}^r Y_i(y) Z_{r+1}^i(x, y), \\ R_{r+1,j} &= \sum_{|i|=0}^{r+1+j} Y_i(y) R_{r+1,j}^i(x, y), \quad (j = 0, 1) \end{aligned}$$

we need to clarify the expression of $Z_{r+1}^i, R_{r+1,j}^i$. Clearly, we have

$$(A.6) \quad Z_{r+1}^i = Z_r^i + [R_{r,0}^i].$$

The expression of $R_{r+1,j}$ is a bit complicated, Towards that, let us do some computation. We note

$$\begin{aligned} \left\langle \nabla h(p) - \omega, \frac{\partial W_r}{\partial x} \right\rangle &= \sum_{j=1}^n \sum_{l=1}^n \frac{\partial^2 h}{\partial x_j \partial x_l} y_l \sum_{i=0}^r Y_i \frac{\partial W_r^i}{\partial x_l}, \\ \{Z_r + R_{r,0}, W_r\} &= \sum_{|i'+i''|\leq r} Y_{i'} Y_{i''} \{Z_r^{i'} + R_{r,0}^{i'}, W_r^{i''}\} \\ &\quad + \sum_{|i'+i''|\leq r} Y_{i''} \left\langle \frac{\partial Y_{i'}}{\partial y}, \frac{\partial W_r^{i''}}{\partial x} \right\rangle (Z_r^{i'} + R_{r,0}^{i'}) \\ &\quad + \sum_{|i'+i''|\leq r} Y_{i'} \left\langle \frac{\partial Y_{i''}}{\partial y}, \frac{\partial (Z_r^{i'} + R_{r,0}^{i'})}{\partial x} \right\rangle W_r^{i''}. \end{aligned}$$

Let $\iota_j \in \mathbb{Z}^n$ denote the vector whose j -th component is 1 and all other components are 0. For $i', i'' \in \mathbb{Z}^n$, the notation $i' \geq i''$ implies that $i'_j \geq i''_j$ holds for each $j \leq n$. For each $i \neq 0$, we define

$$\begin{aligned} \bar{R}_{r+1,0}^i &= \sum_{j=1}^n \sum_{i'+\iota_j=i} \frac{\partial^2 h}{\partial x_j \partial x_l} \frac{\partial W_r^{i'}}{\partial x_j} + \sum_{i'+i''=i} \{Z_r^{i'} + R_{r,0}^{i'}, W_r^{i''}\} \\ &\quad + \sum_{j=1}^n \sum_{i'+i''-\iota_j=i} \left(i'_j (Z_r^{i'} + R_{r,0}^{i'}) \frac{\partial W_r^{i''}}{\partial x_j} + i''_j W_r^{i''} \frac{\partial (Z_r^{i'} + R_{r,0}^{i'})}{\partial x_j} \right) \\ &\quad + \sum_{i' \geq i} \prod_{j=1}^n C_{i'_j}^{i_j} V_{r,j}^{i'_j-i_j} R_{r,1}^{i'} \circ \mathcal{F}_r. \end{aligned}$$

In particular, by the definition, for $r = 0$ we have

$$\begin{aligned} \bar{R}_{r+1,0}^0 &= \{Z_r^0 + R_{r,0}^0, W_r^0\} + R_{r,1}^0 \circ \mathcal{F}_r \\ &\quad + \sum_{j=1}^n W_r^{\iota_j} \frac{\partial (Z_r^0 + R_{r,0}^0)}{\partial x_j} + \sum_{j=1}^n (Z_r^{\iota_j} + R_{r,0}^{\iota_j}) \frac{\partial W_r^0}{\partial x_j}, \end{aligned}$$

for $r+1 < |i| \leq 2r$ we have

$$\begin{aligned} \bar{R}_{r+1,0}^i &= \sum_{j=1}^n \sum_{i'+i''-\iota_j=i} \left(i'_j (Z_r^{i'} + R_{r,0}^{i'}) \frac{\partial W_r^{i''}}{\partial x_j} + i''_j W_r^{i''} \frac{\partial (Z_r^{i'} + R_{r,0}^{i'})}{\partial x_j} \right) \\ &\quad + \sum_{i'+i''=i} \{Z_r^{i'} + R_{r,0}^{i'}, W_r^{i''}\}, \end{aligned}$$

and for $|r| > 2r$ we have

$$\bar{R}_{r+1,0}^i = 0.$$

Now, for those i with $|i| < r+1$, we let

$$R_{r+1,0}^i = \bar{R}_{r+1,0}^i.$$

To define $R_{r+1,0}^i$ for $|i| = r+1$, we introduce some notation more. $i \succ i'$ implies that there exists $j_0 \in \{1, 2, \dots, n\}$ such that $i_j = i'_j$ for each $j < j_0$ and $i_{j_0} > i'_{j_0}$. For each

$i \in \mathbb{Z}_+^n$ let

$$\Lambda_i = \{i' \in \mathbb{Z}^n : i' \geq i\},$$

with which we define

$$\Delta\Lambda_i = \Lambda_i \setminus \bigcup_{\substack{i' \succ i \\ |i'| = r+1}} \Lambda_{i'}.$$

For each $i \in \mathbb{Z}^n$ with $|i| = r+1$ we let

$$R_{r+1,0}^i = \sum_{i' \in \Delta\Lambda_i} Y_{i'-i} \bar{R}_{r+1,0}^{i'}.$$

By the definition, for $i \neq 0$ with $|i| \leq r+1$ we have

$$\|\bar{R}_{r+1,0}^i\|_{k-r-1} \leq \bar{D}_r T (\eta_r \delta_r^{-|i|+1} + \eta_r (\epsilon_r + \eta_r) \delta_r^{-|i|-1}) + \frac{1}{2} \eta_{r+1};$$

for $i \neq 0$ with $r+1 < |i| \leq 2r$ we have

$$\|\bar{R}_{r+1,0}^i\|_{k-r-1} \leq \bar{D}_r T \eta_r (\epsilon_r + \eta_r) \delta_r^{-|i|-1};$$

and for $i = 0$ we have

$$\|\bar{R}_{r+1,0}^0\|_{k-r-1} \leq \bar{D}_r T \epsilon_r \eta_r \delta_r^{-1} + \frac{1}{2} \eta_{r+1},$$

where $\bar{D}_r = \bar{D}_r(n, r, h) > 0$ is a constant. Thus, we find that

$$\eta_{r+1} < D_r (T \eta_r \delta_r + (\epsilon_r + \eta_r) \eta_r \delta_r^{-1})$$

this verifies the second formula in (A.5). One obtains the first formula in (A.5) immediately from (A.6).

To verify the third condition in (A.5), we need to find the expression of term $R_{r+1,1}$. To that goal, we write

$$\{\{H', W_r\}, W_r\} = \sum_{|i| \leq 2r} Y_i(y) \bar{R}_{r+1,1}^i(x, y)$$

where $H' = h + Z_R + R_{r,0}$ and

$$\begin{aligned} \bar{R}_{r+1,1}^i &= \sum_{i'+i''=i} \{\{H, W_r^{i'}\}, W_r^{i''}\} + \sum_{i'+i''=i} i_j'' W_r^{i''} \frac{\partial \{H, W_r^{i'}\}}{\partial x_j} \\ &+ \sum_{i'+i''=i} i_j' \left(\{H, W_r^{i'}\} \frac{\partial W_r^{i''}}{\partial x_j} + \{W_r^{i'}, W_r^{i''}\} \frac{\partial H}{\partial x_j} \right) \\ &+ \sum_{i'+i''=i} i_{j'}' i_{j''}'' W_r^{i''} \left(\frac{\partial^2 H}{\partial x_{j'} \partial x_{j''}} W_r^{i'} + \frac{\partial H}{\partial x_{j'}} \frac{\partial W_r^{i'}}{\partial x_{j''}} \right) \\ &+ \sum_{i'+i''=i} i_{j'}' i_{j''}'' W_r^{i''} \frac{\partial H}{\partial x_{j'}} \frac{\partial W_r^{i''}}{\partial x_{j''}}. \end{aligned}$$

Consequently, one can write

$$R_{r+1,1} = \sum_{|i| \leq r+1} Y_i \circ \phi_{W_r}^{t_0} R_{r+1,1}^i \circ \phi_{W_r}^{t_0}$$

where $t_0 \in (0, 1)$, $R_{r+1,1}^i = \bar{R}_{r+1,1}^i$ for $|i| \leq r+1$ and

$$R_{r+1,0}^i = \sum_{i' \in \Delta_1 \Lambda_i} Y_{i'-i} \bar{R}_{r+1,0}^{i'}, \quad \forall |i| = r+2,$$

with $\Delta_1 \Lambda = \Lambda_i \setminus \cup_{i' \succ i, |i'|=r+2} \Lambda_{i'}$. Therefore, we have

$$\|R_{r+1,1}^i\|_{k-r-2} < D_r T^2 \eta_r^2 \delta_r^{-|i|-1},$$

from which we obtain the third formula in (A.5). \square

We construct a sequence of canonical transformation $\mathcal{F}_r: \mathbb{T}^n \times B_{\delta_{r+1}} \rightarrow \mathbb{T}^n \times B_{\delta_r}$ by using the iterative lemma A.1. Let

$$\epsilon_{r+1} = \epsilon_r + \eta_r, \quad \eta_r = \epsilon^{1+r\sigma}, \quad \delta_r = \left(2 - \frac{r}{k-2}\right) \frac{\mu}{T} \epsilon^\sigma,$$

where $\mu > 0$ to be determined. To apply the iterative lemma, we let δ_r , δ_{r+1} and η_r in (A.4) and in (A.5) be valued as above. Recall that $\varrho = \frac{1}{3}(1 - 3\sigma)$ and let

$$\mu_0 = \min_{r \leq k-2} \frac{k-2}{2D_r(2k-4-r)},$$

and ϵ_0 be the largest ϵ such that the following inequalities hold

$$4D_r(k-2)K_0\epsilon^{\frac{2}{3}-\sigma} \leq \mu, \quad D_r K_0^3 \epsilon^\sigma \leq \mu$$

for $r = 0, 1, \dots, k-2$. We obtain from (A.4) and (A.5) that (A.3) holds for $r+1$ provided it holds for r and $\mu \leq \mu_0$ and $\epsilon \leq \epsilon_0$. Let $\mathcal{F}_\lambda = \mathcal{F}_1 \circ \mathcal{F}_2 \cdots \circ \mathcal{F}_{k-2}$, we obtain from the iterative lemma that $H \circ \mathcal{F}_\lambda$ has the form of (A.1). This completes the proof. \square

There is an analytic version of Theorem A.1, see [Lo]. We introduce a complex domain

$$D(R, \rho, \sigma) = \{(x, y) \in \mathbb{C}^{2n}, \text{dist}(y, B_R) \leq \rho, |\text{Im}x| \leq \sigma\}.$$

when $0 \leq \delta \leq \rho$, $0 \leq \xi \leq \sigma$, we denote the domain $D(R, \rho - \delta, \sigma - \xi)$ by $D - (\delta, \xi)$. Let $\sigma' < 1/3$ and

$$K = K(\epsilon) = K_0 \epsilon^{-\varrho'}, \quad \varrho' = \frac{1}{2}(1 - 3\sigma').$$

Theorem A.2. *The Hamiltonian $H(x, y, t) = h(y) + P_\epsilon(x, y, t)$ is assumed analytic in $D(R, \rho, \sigma)$ and $|P_\epsilon|_D \leq \epsilon$. Some small $\epsilon_0 = \epsilon_0(M, m, n, k) > 0$ exists so that for each $\epsilon \leq \epsilon_0$ and each $y \in B_R$ there exist $y_\lambda \in B_R$ ($\lambda \in \Lambda_{K,R}$) such that $|y - y_\lambda| \leq \mu T^{-1} \epsilon^{\sigma'}$ with $\omega_\lambda = \omega(y_\lambda)$ being rational of period T with $T \leq K$, and a canonical transformation \mathcal{F}_λ well defined on*

$$D_{y_\lambda, \epsilon} = \{(x, y) \in \mathbb{T}^n \times \mathbb{R}^n : |y - y_\lambda| \leq \mu T^{-1} \epsilon^{\sigma'}\}$$

which reduce the Hamiltonian into the normal form

$$H \circ \mathcal{F}_\lambda(x, y, t) = h(y) + Z(x, y, t) + R(x, y, t)$$

where Z is resonant with respect to ω_λ and R is a higher order term comparing with Z :

$$|Z| \leq 2\epsilon, \quad |R| \leq \epsilon e^{-\epsilon^{\sigma'}} \quad \text{in } D_{y_\lambda, \epsilon} \times \mathbb{T}.$$

A.2. Covering property. Recall that the set of frequencies $\{\omega_\lambda : \lambda \in \Lambda_{K,R}\} \subset B_{MR}$ each of which is rational of period T with $T \leq K$, and the domains $\{D_{y_\lambda, \epsilon} : \lambda \in \Lambda_{K,R}\}$ where the iteration of KAM is carried (see Theorem A.1 for definition).

Theorem A.3. *The following covering property holds*

$$(A.7) \quad \bigcup_{\lambda \in \Lambda_{K,R}} \mathcal{F}_\lambda^{-1} D_{y_\lambda, \epsilon} \supset \mathbb{T}^n \times B_R \quad \text{provided} \quad \sigma < \frac{1}{3n+3}.$$

Proof. To show the covering property, we use Dirichlet's approximation theorem. For real x one has

$$x = [x] + \{x\},$$

where $[x] \in \mathbb{Z}$ the integer part, and $\{x\} \in (0, 1)$. We use notation

$$\|x\|_{\mathbb{Z}} = \inf\{\{x\}, 1 - \{x\}\} = \text{dist}(x, \mathbb{Z}).$$

If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ one sets

$$\|x\|_{\mathbb{Z}} = \sup_{i=1,2,\dots,n} \|x_i\|_{\mathbb{Z}}.$$

Proposition A.1. (Dirichlet, see for examples, [Cas], [Sch]) *Let $\omega \in \mathbb{R}^n$ and K a real number with $K > 1$. There exists an integer k , $1 \leq k < K$, such that*

$$\|k\omega\|_{\mathbb{Z}} \leq K^{-\frac{1}{n}}.$$

For any $\omega \in \mathbb{R}^n$, by applying Dirichlet's theorem, we find some rational vector ω^* existing such that $K'\omega^* \in \mathbb{Z}^n$ with $K' \leq K$ and

$$\text{dist}(K'\omega, K'\omega^*) \leq \sqrt{n}K^{-\frac{1}{n}},$$

here, ω^* is a rational vector of period T . Since h is assumed strictly convex, there exist two points $y, y^* \in B_R$ such that $\nabla h(y) = \omega$, $\nabla h(y^*) = \omega^*$ and

$$\text{dist}(p, p^*) \leq \frac{1}{m} \text{dist}(\omega, \omega^*).$$

The condition $\|y - y^*\| \leq \mu T^{-1} \epsilon^\sigma$ is guaranteed if we choose

$$(A.8) \quad K^{\frac{1}{n}} = \frac{\sqrt{n}}{\mu m} \epsilon^{-\sigma}.$$

As $T \leq K$ is required, the following should be satisfied:

$$K \leq K_0 \epsilon^{-\frac{1}{3}(1-3\sigma)},$$

that is, referring to (A.8),

$$\epsilon^{\frac{1-3\sigma}{3} - n\sigma} \leq \left(\frac{\mu m}{\sqrt{n}}\right)^n \frac{1}{K_0},$$

which determines a threshold for ϵ provided:

$$\sigma < \frac{1}{3n+3}.$$

As we choose σ satisfying this condition, the covering property (A.7) is proved. \square

For the purpose of this paper, the covering property (A.7) for the whole space is not necessary, instead,

Denote by $\mathbf{k} = (k_1, \dots, k_{n-1})$ a $n \times (n-1)$ matrix, where k_1, \dots, k_{n-1} are integer vectors. We consider the $n-1$ resonance line

$$\Gamma_{\mathbf{k}} = \{y \in \mathbb{R}^n : \langle k_i, \partial h(y) \rangle = 0 \ \forall i = 1, \dots, n-1\}.$$

If the covering property (A.7) in Theorem A.1 is replaced by covering a neighborhood of the line

$$(A.9) \quad \bigcup_{\lambda \in \Lambda_{K,R}} D_{y_{\lambda}, \epsilon} \supset \mathbb{T}^n \times \{\|y - y_0\| < \mu K^{-1} \epsilon^\sigma : y_0 \in \Gamma_{\mathbf{k}} \cap B_R\}$$

then it works if

$$\sigma < \frac{1}{6}.$$

Indeed, as all frequencies are on a $(n-1)$ -resonance line, by using Dirichlet approximation theorem (Proposition A.1) for $n=1$ we obtain a threshold $\sigma < 1/6$.

Recall that the term Z in (A.1) is resonant with respect to ω , some rational frequency of period $T \leq K$, namely, it has the form

$$Z(x, y) = \sum_{\langle k, \omega \rangle = 0} Z_k(y) e^{i\langle k, x \rangle}.$$

Note that $T\omega$ is an indivisible integer vector, i.e. $\mu T\omega \notin \mathbb{Z}^n$ for any $\mu \in (0, 1)$. There are $n-1$ integer vectors I_2, I_3, \dots, I_n such that the matrix (I_2, I_3, \dots, I_n) is indivisible, $\text{rank}(I_2, I_3, \dots, I_n) = n-1$ and $\langle I_i, \omega \rangle = 0$ holds for each $i \in \{2, 3, \dots, n\}$. Clearly, there is another integer vector I_1 such that the matrix $I = (I_1, I_2, \dots, I_n)$ is uni-module. Obviously, each $k \in \mathbb{Z}^n$ with $\langle k, \omega \rangle = 0$ determines uniquely an integer vector $\bar{k} \in \mathbb{Z}^{n-1}$ such that $k = \sum_{j=1}^{n-1} \bar{k}_j I_{j+1}$.

We introduce a coordinate transformation: $(x, y) \rightarrow (p, q)$ such that

$$(A.10) \quad \tilde{q} = I^t x, \quad \tilde{p} = I^{-1} y.$$

This coordinate transformation is symplectic, $H(I^{-t} \tilde{q}, I \tilde{p})$ is also a function defined in \mathbb{T}^n with respect to \tilde{q} . Let y be the point where $\nabla h(y) = \omega$, then the gradient of $\tilde{h}(\tilde{p}) = h(I \tilde{p})$ satisfies

$$\tilde{\omega} = \nabla \tilde{h}(\tilde{p}) = (0, \dots, 0, \tilde{\omega}_n),$$

and $Z(I \tilde{p}, I^{-t} \tilde{q})$ is independent of q_n , thus we can write $\tilde{Z}(\tilde{p}, \tilde{q}) = \tilde{Z}(p, q, p_n)$ if we use the notation $\tilde{p} = (p, p_n)$ and $\tilde{q} = (q, q_n)$.

Let us consider the time-1-periodically non-autonomous case. Assume $T(\omega, 1) \in \mathbb{Z}^{n+1}$ is an indivisible integer vector. As Z is resonant with respect to ω , we have

$$Z(x, y, t) = \sum_{\langle k, \omega \rangle + l = 0} Z_{k,l}(y) e^{i\langle k, x \rangle + lt}.$$

Thus, there are n integer vectors $I_1, \dots, I_n, J \in \mathbb{Z}^n$ such that $\langle I_i, \omega \rangle + J_i = 0$ for each $i \in \{1, \dots, n\}$. For each $(k, l) \in \mathbb{Z}^{n+1}$ with $\langle k, \omega \rangle + l = 0$, there is uniquely determined $(\bar{k}_1, \dots, \bar{k}_n) \in \mathbb{Z}^n$ such that

$$(k, l) = \sum_{i=1}^n \bar{k}_i (I_i, J_i).$$

By suitably choosing I_i , we can make $I = (I_1, I_2, \dots, I_n)$ be uni-module. Introduce the coordinate transformation (A.10), let y be the point where $\nabla h(y) = \omega$, then the gradient of $\bar{h}(p) = h(Ip)$ satisfies

$$\bar{\omega} = \nabla \bar{h}(p) = -J.$$

Note that each (k, l) with $\langle k, \omega \rangle + l = 0$ uniquely determines $\bar{k} \in \mathbb{Z}^n$ such that $(k, l) = \bar{k}(I^t, J)$. As we have

$$Z(I^{-t}q, Ip, t) = \sum_{\bar{k} \in \mathbb{Z}^n} Z_{k,l}(Ip) e^{i\langle \bar{k}, q + Jt \rangle},$$

in the new coordinates the resonant term $\bar{Z} = \bar{Z}(p, q + Jt)$. Let $q' = q + Jt, p' = p$ and let $h'(p') = \bar{h}(p') + \langle J, p \rangle$, we find the Hamiltonian equation of $h'(p') + \bar{Z}(p', q')$ is the same as the Hamiltonian equation of $\bar{h}(p) + \bar{Z}(p, q + Jt)$.

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